BROCCOLI CURVES AND THE TROPICAL INVARIANCE OF WELSCHINGER NUMBERS

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ABSTRACT. In this paper we introduce broccoli curves, certain plane tropical curves of genus zero related to real algebraic curves. The numbers of these broccoli curves through given points are independent of the chosen points — for arbitrary choices of the directions of the ends of the curves, possibly with higher weights, and also if some of the ends are fixed. In the toric Del Pezzo case we show that these broccoli invariants are equal to the Welschinger invariants (with real and complex conjugate point conditions), thus providing a proof of the independence of Welschinger invariants of the point conditions within tropical geometry. The general case gives rise to a tropical Caporaso-Harris formula for broccoli curves which suffices to compute all Welschinger invariants of the plane.

1. INTRODUCTION

1.1. **Background on tropical Welschinger numbers.** Welschinger invariants of real toric unnodal Del Pezzo surfaces count real rational curves belonging to an ample linear system D and passing through a generic conjugation invariant set $\mathscr{P}$ of $-K_\Sigma \cdot D - 1$ points, weighted with ± 1, depending on the nodes of the curve. It was shown in [Wel03] resp. [Wel05] that these numbers are invariant, i.e. do not depend on the choice of $\mathscr{P}$. They can be thought of as real analogues of the numbers of complex rational curves belonging to a fixed linear system and satisfying point conditions, which in the case of $\mathbb{P}^2$ are the genus-0 Gromov-Witten invariants.

By Mikhalkin's Correspondence Theorem [Mik05], Gromov-Witten invariants of the plane (resp. the complex enumerative numbers of other toric surfaces) can be determined via tropical geometry, by counting tropical curves of a fixed degree and satisfying point conditions. Each tropical curve has to be counted with a "complex multiplicity" which reflects how many complex curves map to it under tropicalization.

Welschinger invariants can be computed via tropical geometry in a similar way: one can define a certain count of tropical curves and prove a Correspondence Theorem stating that this tropical count equals the Welschinger invariant. For the case when $\mathscr{P}$ consists of only real points, such a Correspondence Theorem is proved in [Mik05], the general case is proved in [Shu06].

If $\mathscr{P}$ consists of only real points, the tropical curves we have to count to get Welschinger invariants are exactly the same as the ones we need to count to determine complex enumerative numbers — we just have to count them with a different, "real" multiplicity. The lattice path algorithm of [Mik05] enumerates the tropical curves we have to count. If $\mathscr{P}$ also contains pairs of complex conjugate points, we have to count tropical curves satisfying some more special conditions. The lattice path algorithm is generalized in [Shu06] to an algorithm that computes the corresponding Welschinger invariants.

It follows from the Correspondence Theorem and the fact that Welschinger invariants are independent of the point conditions that the corresponding tropical count is also invariant, i.e. does not depend on the position of the points that we require the tropical curves to pass through.

Still, it is interesting to find an argument within tropical geometry that proves the invariance of the tropical numbers. For the case when $\mathscr{P}$ consists of only real points, such a statement follows easily since the corresponding tropical count can be shown to be locally invariant, i.e. invariant around a codimension-1 cone of the corresponding moduli space of curves. In addition, such a codimension-1 cone is specified by a 4-valent vertex of a tropical curve, and it is sufficient to consider the curves locally around this 4-valent vertex. This tropical invariance statement was proved in [IKS09], and generalized to a relative situation where we count tropical curves with ends of higher weights with their real multiplicity. In [GM07a], tropical curves with ends of higher weights counted with their complex multiplicity are shown to determine relative Gromov-Witten invariants of the plane, i.e. numbers of complex plane curves satisfying point conditions and tangency conditions to a given line L. Thus one could imagine that the tropical relative real count corresponds to numbers of real curves satisfying point and tangency conditions. This is not true however since such a real count is not invariant. The tropical proof of the invariance in this situation thus led to the construction of new tropical invariant numbers whose real counterparts are yet to be understood.

Also, because of the invariance of the tropical relative real count one can establish a Caporaso-Harris formula for Welschinger invariants for which $\mathscr{P}$ consists of only real points. Originally, Caporaso and Harris developed their algorithm to determine the numbers of complex curves satisfying point conditions [CH98]. They defined the above mentioned relative Gromov-Witten invariants and specialized one point after the other to lie on the line L. Since a curve of degree d intersects L in d points, after some steps the curves become reducible and the line L splits off as a component. One then collects the contributions from all the components and thus produces recursive relations among the relative Gromov-Witten invariants that finally suffice to compute the numbers of complex curves satisfying point conditions. A tropical counterpart of this algorithm has been established in [GM07a]. There, one moves one point after the other to the far left part of the plane (but still in general position). The tropical curves then do not become reducible, but in a sense decompose into two parts, leading to recursive relations. The left part, passing through the moved point, is called a floor [BM08]. In [IKS09] the authors use the same idea to specialize points and consider tropical curves decomposing into a floor and another part, only now they have to deal with the real multiplicity for these tropical curves. The formula one thus obtains computes tropical Welschinger numbers which are equal to their classical counterparts by the Correspondence Theorem. Since this formula is recursive it is much more efficient for the computation of Welschinger invariants than the lattice path algorithm mentioned above. The lattice path algorithm and the Caporaso-Harris formula are currently the only known methods to compute Welschinger invariants. There is work in progress however to compute Welschinger invariants without tropical methods [Sol].

Now let us discuss the situation when $\mathscr{P}$ does not only contain real points, but also pairs of complex conjugate points. As already mentioned, also here a Correspondence Theorem exists to relate these Welschinger invariants to a certain count of tropical curves, and one can count the tropical curves with a generalized lattice path algorithm [Shu06]. In addition, it follows of course again from the Correspondence Theorem together with the Welschinger Theorem that the tropical count is invariant. However, the tropical count is no longer locally invariant in the moduli space, and thus there was no known tropical proof for the (global) invariance of the tropical count. Even worse, if we try to generalize the tropical count to relative numbers, i.e. to curves with ends of higher weight, then these numbers are no longer invariant. However, one can still pick a special configuration of points, namely the result after applying the Caporaso-Harris algorithm as many times as possible. Then each point is followed by a point which is far more left, and the curves totally decompose into floors. They can then be counted by means of floor diagrams. Although the tropical relative count is not invariant, the floor diagram count leads to a Caporaso-Harris type formula which is sufficient to compute all Welschinger invariants of the plane [ABLdM10].
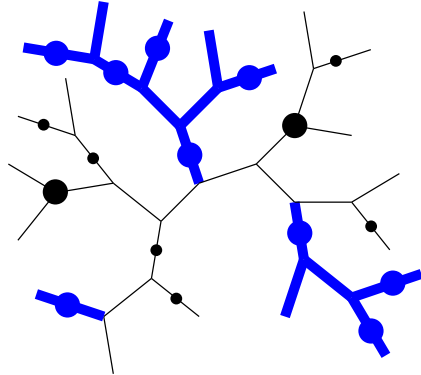
1.2. **The content of this paper.** The aim of this paper is to give a tropical proof of the invariance of tropical Welschinger numbers for real and complex conjugate points. As an additional result this

will allow us to construct corresponding tropical invariants in the relative setting (or more generally for any choice of directions for the ends of the curve). Using this result, we can then establish a Caporaso-Harris formula for rational curves in a much simpler way than in [ABLdM10].
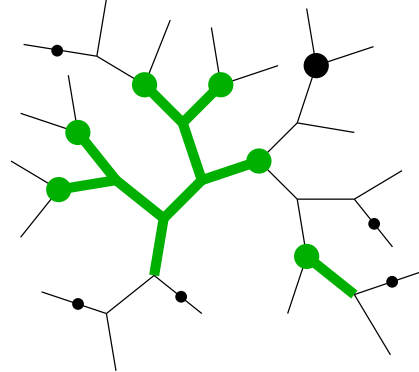
The key idea to achieve this is to modify (and in fact also simplify) the class of tropical curves that we count in order to obtain the invariants. This modification is small enough so that the (weighted) number of these curves through given points remains the same in the toric Del Pezzo case, but big enough so that their count becomes locally invariant in the moduli space.

Let us explain this modification in more detail. For this it is important to distinguish between odd and even edges of a tropical curve, i.e. edges whose weight is odd resp. even. In our pictures we will always draw odd edges as thin lines and even edges as thick lines. Moreover, we will draw real points as thin dots and complex points (i.e. those corresponding to a pair of complex conjugate points in the algebraic case) as thick dots. All our curves will be of genus zero.

The tropical curves that are usually counted to obtain the Welschinger invariants — we will call them Welschinger curves — then have the property that each connected component of even edges is connected to the rest of the curve at exactly one point (we can think of such a component as an end tree). Moreover, real points cannot lie on end trees, and each complex point is either on an end tree or at a 4-valent vertex [Shu06]. Below on the left we have drawn a typical (schematic) picture of such a Welschinger curve, with the end trees marked blue. Note that the marking lying on a point is itself an edge, so that the 4-valent complex markings away from the end trees look like 3-valent vertices in the picture.
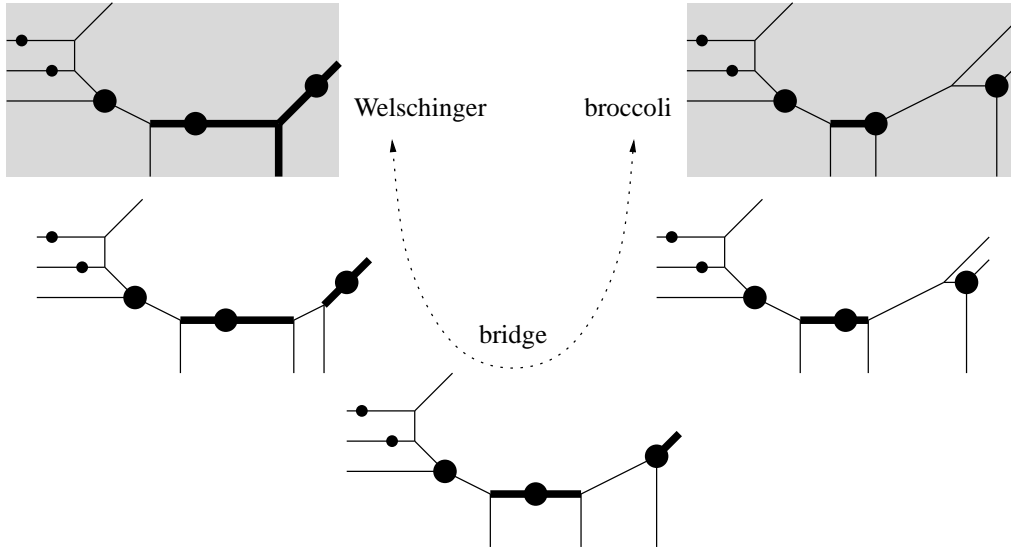

A Welschinger curve   A broccoli curve

We now change this condition slightly to obtain a different class of curves that we call broccoli curves: each connected component of even edges can now be connected to the rest of the curve at several points, of which exactly one is a 3-valent vertex without marking as before (the "broccoli stem"), and the remaining ones are complex points (the "broccoli florets"). The even part of the curve (the "broccoli part") may not contain any points in its interior, whereas away from this part we can have real points at 3-valent and complex points at 4-valent vertices as before. The picture above on the right shows a typical schematic example of a broccoli curve, with the broccoli part drawn in green. Note that, in contrast to Welschinger curves, complex points are always at 4-valent vertices in broccoli curves.

Broccoli curves have the advantage that their count (with suitably defined multiplicities) is locally invariant in the moduli space, similarly to the situation mentioned above when we count complex curves or Welschinger curves through only real points. Hence counting these curves we obtain well-defined broccoli invariants — even for curves with directions of the ends for which the corresponding Welschinger count would not be invariant of the position of the points.

In addition, we show that in the toric Del Pezzo case broccoli invariants equal Welschinger numbers, thereby giving a new and entirely tropical proof of the invariance of Welschinger numbers. We prove this by constructing bridges between broccoli curves and Welschinger curves which show that their numbers must be equal. To illustrate this concept of bridges in an easy example we have drawn in the picture below a Welschinger curve (which is not a broccoli curve) and a broccoli curve (which is not a Welschinger curve) of degree 3 through the same two real and three complex points. They can be connected by the bridge drawn below those curves: starting from the Welschinger curve we first split the vertical end of weight 2 into two edges of weight 1 until the rightmost complex point becomes 4-valent (in the picture at the bottom), and then split the other end of weight 2 in a similar way until we arrive at the broccoli curve.

It should be noted that this example is a particularly simple bridge as it connects a Welschinger curve to a unique corresponding broccoli curve. In general, traversing bridges will involve creating and resolving higher-valent vertices of curves along 1-dimensional families — and as there are usually several possibilities for such resolutions this means that bridges may ramify on their way from the Welschinger to the broccoli side. Bridge curves will be assigned a multiplicity (in a similar way as for Welschinger and broccoli curves), and at each point of the bridge it is just the weighted number of incoming Welschinger and outgoing broccoli curves that is the same — not necessarily the absolute number of them. In particular, bridges do in general not provide a bijection between Welschinger and broccoli curves, in fact not even a well-defined map in either direction.

Another technical thing to note is that we have twice split an even end of weight 2 into two odd ends of weight 1 on the bridge above. This might look like a discontinuous change in the underlying graph of the tropical curve. In order to avoid this inconvenience we will usually parametrize even ends of Welschinger curves as two ends of half the weight (which we call double ends). This way no further end splitting takes place on bridges.

It would certainly be very interesting to see if one could prove a Correspondence Theorem for broccoli curves that relates these tropical curves directly to certain real algebraic ones. So far there is no such statement known; in particular there is no algebraic counterpart to broccoli invariants for directions of the ends of the curves when the corresponding Welschinger number is not an invariant.

This paper is organized as follows. In section 2 we review basic notions of tropical curves and their moduli spaces. In particular, we introduce the notion of oriented curves (i.e. tropical curves with the edges oriented in a certain way), a tool which simplifies proofs in the rest of the paper. The next three sections are dedicated to the different kinds of tropical curves mentioned above: section 3 deals with

broccoli curves; the main result here is theorem 3.6 which states that the counts of broccoli curves do not depend on the position of the points. In a very analogous way, section 4 considers Welschinger curves and shows that their counts yield the Welschinger invariants. We then introduce bridge curves in section 5 and use them in corollary 5.16 to prove that Welschinger and broccoli invariants agree in the toric Del Pezzo case, and thus that the Welschinger invariants then do not depend on the choice of point conditions (corollary 5.17). Finally, the existence of well-defined broccoli invariants also in the relative case enables us to prove a Caporaso-Harris formula for Welschinger invariants of the plane in section 6.

2. ORIENTED MARKED CURVES

Let us start by introducing the tropical curves that we will deal with in this paper. As all our curves will be tropical we usually drop this attribute in the notation. All curves will be in $\mathbb{R}^2$ (parametrized and labeled in the sense of [GKM09] section 4), connected, and of genus 0. Let us quickly recall the definition of these tropical curves, already making the distinction between real and complex markings resp. odd and even edges that we will later need to consider real enumerative invariants.

Definition 2.1 (Marked curves). Let $r,s \in \mathbb{N}$. An (r,s)-*marked (plane tropical) curve* is a tuple $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h)$ for some $n \in \mathbb{N}$ such that:

(a) Γ is a connected rational metric graph, with unbounded edges (with no vertex there) allowed, and such that each vertex has valence at least 3. The unbounded edges of Γ will be called the *ends* of C.

(b) $h : \Gamma \to \mathbb{R}^2$ is a continuous map that is integer affine linear on each edge of Γ, i.e. on each edge E it is of the form $h(t) = a + tv$ for some $a \in \mathbb{R}^2$ and $v \in \mathbb{Z}^2$. If we parameterize E starting at the vertex $V \in \partial E$ the vector v in this equation will be denoted $v(E,V)$ and called the *direction (vector)* of E starting at V. For an end E we will also write $v(E)$ instead of $v(E,V)$, where V is the unique vertex of E. We say that an edge is *contracted* if its direction is 0.

(c) At each vertex V of Γ the *balancing condition*
$$\sum_{E:V \in \partial E} v(E,V) = 0$$
holds.

(d) $x_1, \ldots, x_{r+s}$ is a labeling of the contracted ends, $y_1, \ldots, y_n$ a labeling of the non-contracted ends of C. We call $x_1, \ldots, x_{r+s}$ the *markings* or *marked ends*; more specifically the r ends $x_1, \ldots, x_r$ are called the *real markings*, the s ends $x_{r+1}, \ldots, x_{r+s}$ the *complex markings* of C. The other ends $y_1, \ldots, y_n$ are called the *unmarked ends*; the collection $(v(y_1), \ldots, v(y_n))$ of their directions will be called the *degree* $\Delta = \Delta(C)$ of C. We denote the number n of vectors in Δ by $|\Delta|$.

The space of all (r,s)-marked curves of degree Δ will be denoted $M_{(r,s)}(\Delta)$.
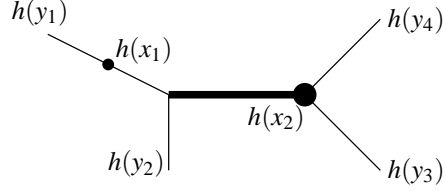
Definition 2.2 (Even and odd edges, weights). Let C be a marked curve.

(a) A vector in $\mathbb{Z}^2$ will be called *even* if both its coordinates are even, and *odd* otherwise. We say that an edge of C is even resp. odd if its direction vector is even resp. odd.

(b) If we write the direction vector of an edge E of C as a non-negative multiple $\omega(E)$ of a primitive integral vector we call this number $\omega(E)$ the *weight* of E. Note that E is even resp. odd if and only if its weight is even resp. odd.

Convention 2.3. When drawing a marked curve $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h)$ we will usually only show the image $h(\Gamma) \subset \mathbb{R}^2$, together with the image points $h(x_1), \ldots, h(x_{r+s})$ of the markings. These image points will be drawn as small dots for real markings and as big dots for complex markings. The other edges will always be displayed as thin lines for odd edges and as thick lines for even edges. Unmarked contracted edges would not be visible in these pictures, but (although allowed) they will not play a special role in this paper.

Example 2.4. Using convention 2.3, the picture on the right shows a $(1,1)$-marked plane curve of degree $((-2,1),(0,-1),(1,-1),(1,1))$. It has two 3-valent vertices and one 4-valent vertex. The thick edge has direction $(-2,0)$ starting at the complex marking. For clarity we have labeled all the ends in the picture, but in the future we will usually omit this as the actual labeling will not be relevant for most of our arguments.

Remark 2.5. Note that our moduli space $M_{(r,s)}(\Delta)$ is precisely the space $\mathscr{M}^{\text{lab}}_{0,r+s,\text{trop}}(\mathbb{R}^2, \Delta)$ of $(r+s)$-marked plane labeled tropical curves of [GKM09] definition 4.1. As such it is a polyhedral complex, and in fact even a tropical variety (see [GKM09] proposition 4.7). In this paper we will not need its structure as a tropical variety however, but only consider $M_{(r,s)}(\Delta)$ as an abstract polyhedral complex with polyhedral structure induced by the combinatorial types of the curves. Let us quickly establish this notation.

Definition 2.6 (Combinatorial types). Let $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h) \in M_{(r,s)}(\Delta)$ be a marked curve. The *combinatorial type* of C is the data of the non-metric graph Γ, together with the labeling $x_1, \ldots, x_{r+s}, y_1, \ldots, y_n$ of the ends and the directions of all edges. For such a combinatorial type α we denote by $M^\alpha_{(r,s)}(\Delta)$ the subspace of $M_{(r,s)}(\Delta)$ of all marked curves of type α.

Remark 2.7 ($M_{(r,s)}(\Delta)$ as a polyhedral complex). In the same way as in [GM08] example 2.13 the moduli spaces $M_{(r,s)}(\Delta)$ are abstract polyhedral complexes in the sense of [GM08] definition 2.12, i.e. they can be obtained by glueing finitely many real polyhedra along their faces. The open cells of these complexes are exactly the subspaces $M^\alpha_{(r,s)}(\Delta)$, where α runs over all combinatorial types of curves in $M_{(r,s)}(\Delta)$. The curves in such a cell (i.e. for a fixed combinatorial type) are parametrized by the position in $\mathbb{R}^2$ of a chosen root vertex and the lengths of all bounded edges (which need to be positive). Hence $M^\alpha_{(r,s)}(\Delta)$ can be thought of as an open polyhedron whose dimension is equal to 2 plus the number of bounded edges in the combinatorial type α. We will call this dimension the *dimension* $\dim \alpha$ of the type α.

Let us now consider enumerative questions for our curves. In addition to the usual incidence conditions we want to be able to require that some of the unmarked ends are fixed, i.e. map to a given line in $\mathbb{R}^2$. To count such curves we will now introduce the corresponding evaluation maps. Moreover, to be able to compensate for the overcounting due to the labeling of the non-fixed unmarked ends we will define the group of permutations of these ends that keep the degree fixed.

Definition 2.8 (Evaluation maps and $G(\Delta, F)$). Let $r, s \geq 0$, let $\Delta = (v_1, \ldots, v_n)$ be a collection of vectors in $\mathbb{Z}^2 \setminus \{0\}$, and let $F \subset \{1, \ldots, n\}$.

(a) The *evaluation map* ev_F (with *set of fixed ends* F) on $M_{(r,s)}(\Delta)$ is defined to be

$$\text{ev}_F : \quad M_{(r,s)}(\Delta) \longrightarrow (\mathbb{R}^2)^{r+s} \times \prod_{i \in F} \left(\mathbb{R}^2 / \langle v_i \rangle\right) \quad \cong \mathbb{R}^{2(r+s)+|F|}$$

$$(\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h) \longmapsto \left((h(x_1), \ldots, h(x_{r+s})), (h(y_i) : i \in F)\right).$$

In our pictures we will indicate ends that we would like to be considered fixed with a small orthogonal bar at the infinite side.

(b) We denote by $G(\Delta, F)$ the subgroup of the symmetric group $\mathbb{S}_n$ of all permutations such that $\sigma(i) = i$ for all $i \in F$ and $v_{\sigma(i)} = v_i$ for all $i = 1, \ldots, n$.

For the case $F = \emptyset$ of no fixed ends we denote ev_F simply by ev and $G(\Delta, F)$ by $G(\Delta)$.

Remark 2.9. As in [GM08] example 3.3 these evaluation maps are morphisms of polyhedral complexes in the sense that they are continuous maps that are linear on each cell $M_{(r,s)}^{\alpha}(\Delta)$ of $M_{(r,s)}(\Delta)$. Note that $G(\Delta, F)$ acts on $M_{(r,s)}(\Delta)$ by permuting the unmarked ends, and that ev_F is invariant under this operation. By definition, if

$$\mathscr{P} = \left((P_1, \ldots, P_{r+s}), (Q_i : i \in F)\right) \in (\mathbb{R}^2)^{r+s} \times \prod_{i \in F} \left(\mathbb{R}^2 / \langle v_i \rangle\right)$$

then the inverse image $\mathrm{ev}_F^{-1}(\mathscr{P})$ consists of all $(r+s)$-marked curves $(\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h)$ of degree Δ that pass through $P_i \in \mathbb{R}^2$ at the marked point x_i for all $i = 1, \ldots, r+s$ and map the i-th unmarked end y_i to the line $Q_i \in \mathbb{R}^2 / \langle v_i \rangle$ for all $i \in F$. We call $\mathscr{P}$ a *collection of conditions* for ev_F.

Of course, when counting curves we must assume that the conditions we impose are in general position so that the dimension of the space of curves satisfying them is as expected. Let us define this notion rigorously.

Definition 2.10 (General and special position of points). Let $N \in \mathbb{N}$, and let $f : M \to \mathbb{R}^N$ be a morphism of polyhedral complexes (as e.g. the evaluation map ev_F of definition 2.8 (a)). Then the union $\bigcup_\alpha f(M^\alpha) \subset \mathbb{R}^N$, taken over all cells M^α of M such that the polyhedron $f(M^\alpha)$ has dimension at most $N-1$, is called the locus of points *in special position* for f. Its complement is denoted the locus of points *in general position* for f.

Remark 2.11. Note that the locus of points in general position for a morphism $f : M \to \mathbb{R}^N$ is by definition the complement of finitely many polyhedra of positive codimension in $\mathbb{R}^N$. In particular, it is a dense open subset of $\mathbb{R}^N$.

Example 2.12. Let $M \subset M_{(r,s)}(\Delta)$ be a polyhedral subcomplex, and let $F \subset \{1, \ldots, |\Delta|\}$. Then a collection of conditions $\mathscr{P} \in \mathbb{R}^{2(r+s)+|F|}$ as in remark 2.9 is in general position for $\mathrm{ev}_F : M \to \mathbb{R}^{2(r+s)+|F|}$ if and only if for each curve in M satisfying the conditions $\mathscr{P}$ and every small perturbation of these conditions we can still find a curve of the same combinatorial type satisfying them.
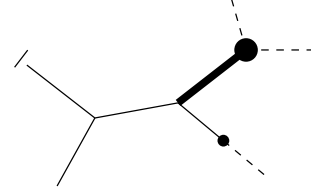
Collections of conditions in general position for the evaluation map have a special property that will be crucial for the rest of the paper: in [GM08] remark 3.7 it was shown that every 3-valent curve $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h) \in M_{(r,s)}(\Delta)$ through a collection of $r+s = |\Delta| - 1$ points in general position for the evaluation map $\mathrm{ev} : M_{(r,s)}(\Delta) \to \mathbb{R}^{2(r+s)}$ without fixed ends has the property that each connected component of $\Gamma \backslash (x_1 \cup \cdots \cup x_{r+s})$ contains exactly one unmarked end. For the purposes of this paper we need the following generalization of this statement to curves that are not necessarily 3-valent and evaluation maps that may have fixed ends.

Lemma 2.13. *Let $M \subset M_{(r,s)}(\Delta)$ be a polyhedral subcomplex, and let $\mathscr{P}$ be a collection of conditions in general position for the evaluation map $\mathrm{ev}_F : M \to \mathbb{R}^{2(r+s)+|F|}$. Consider a curve $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h) \in \mathrm{ev}_F^{-1}(\mathscr{P})$ satisfying these conditions. Then:*

(a) *Each connected component of $\Gamma \backslash (x_1 \cup \cdots \cup x_{r+s})$ has at least one unmarked end y_i with $i \notin F$.*

(b) *If the combinatorial type of C has dimension $2(r+s) + |F|$ and every vertex of C that is not adjacent to a marking is 3-valent then every connected component of $\Gamma \backslash (x_1 \cup \cdots \cup x_{r+s})$ as in (a) has exactly one unmarked end y_i with $i \notin F$.*

Proof. Consider a connected component of $\Gamma \backslash (x_1 \cup \cdots \cup x_{r+s})$ and denote by Γ' its closure in Γ. We can consider Γ' as a graph, having a certain number a of unbounded fixed ends, b unbounded non-fixed ends, and c bounded ends (i.e. 1-valent vertices) at markings of C. The statement of part (a) of the lemma is that $b \geq 1$, with equality holding in case (b). For an example, in the picture below on the right Γ' consists of the solidly drawn lines; the curve continues in some way behind the dashed lines. Recall that fixed ends are indicated by small bars at the infinite sides. Hence in our example we have $a = 1$, $b = 1$, and $c = 2$.
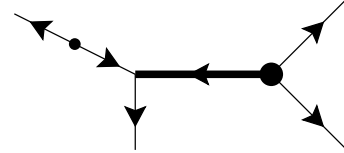
By the same argument as in remark 2.7, the graph Γ' as well as the map $h|_{\Gamma'}$ is fixed by the position of a root vertex in Γ' and the lengths of all bounded edges of Γ'. But an easy combinatorial argument shows that the number of bounded edges of Γ' is equal to $a + b + 2c - 3 - \sum_V (\text{val} V - 3)$, with the sum taken over all vertices V that are not adjacent to a marking. Hence Γ' and its image $h|_{\Gamma'}$ can vary with $a + b + 2c - 1 - \sum_V (\text{val} V - 3)$ real parameters in M.

On the other hand, Γ' together with $h|_{\Gamma'}$ fixes $a + 2c$ coordinates in the image of the evaluation map, namely the positions of the a fixed ends and the c markings in Γ'.

Hence $b = 0$ is impossible: then these $a + 2c$ coordinates of the evaluation map would vary with fewer than $a + 2c$ coordinates of M, meaning that the image of ev_F on the cell of C cannot be full-dimensional and thus $\mathscr{P}$ cannot have been in general position. This proves (a). But in case (b) $b > 1$ is impossible as well: then by assumption we have $\text{val} V = 3$ for all V as above, and thus one could fix a position for the fixed ends and markings at Γ' in $\mathbb{R}^2$ and still obtain a $(b-1)$-dimensional family for Γ' and $h|_{\Gamma'}$. As a movement in this family does not change anything away from Γ' this means that ev_F is not injective on the cell of M corresponding to C. But ev_F is surjective on this cell as $\mathscr{P}$ is in general position. This is a contradiction since by assumption the source and the target of the restriction of ev_F to the cell corresponding to C have the same dimension. $\qquad \square$

Remark 2.14. The important consequence of lemma 2.13 (b) is that — whenever it is applicable — it means that there is a unique way to orient every unmarked edge of $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h)$ so that it points towards the unique unmarked non-fixed end of the component of $\Gamma \backslash (x_1 \cup \cdots \cup x_{r+s})$ containing the edge. The picture on the right shows this for the curve of example 2.4. Note that the arrow will always point inwards on fixed ends, and outwards on non-fixed ends.
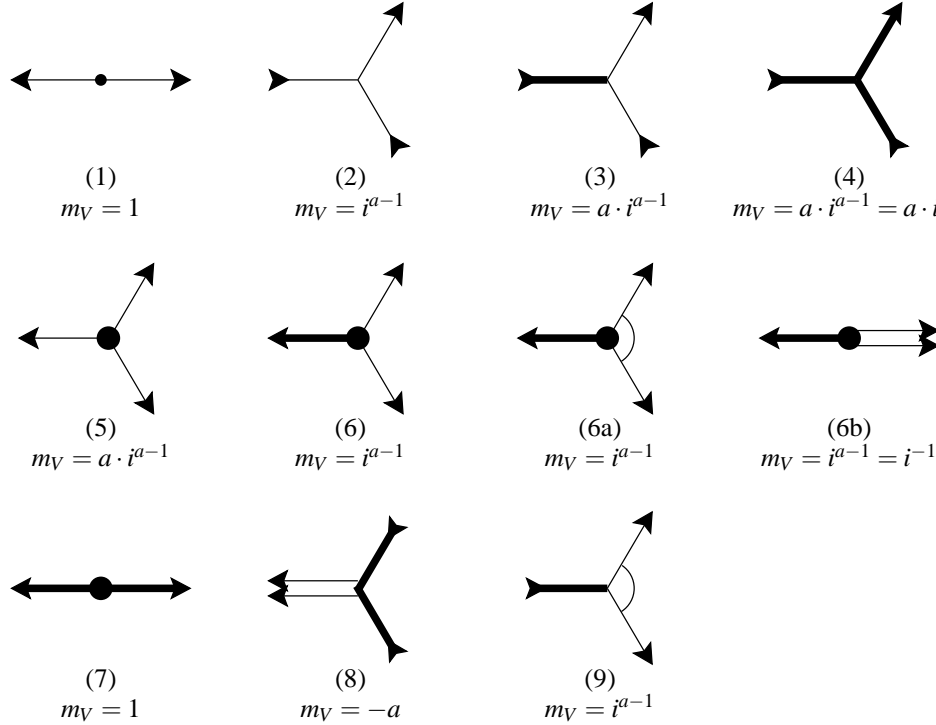
To be able to talk about this concept in the future we will now introduce the notion of oriented curves.

Definition 2.15 (Oriented marked curves). An *oriented (r,s)-marked curve* is an (r,s)-marked curve $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h)$ as in definition 2.1 in which each unmarked edge of Γ is equipped with an orientation (which we will draw as arrows in our pictures). In accordance with our above idea, the subset $F = F(C) \subset \{1, \ldots, n\}$ of all i such that the unmarked end y_i is oriented inwards is called the *set of fixed ends* of C. The space of all oriented (r,s)-marked curves with a given degree Δ and set of fixed ends F will be denoted $M^{\text{or}}_{(r,s)}(\Delta, F)$; for the case $F = \emptyset$ of no fixed ends we write $M^{\text{or}}_{(r,s)}(\Delta, \emptyset)$ also as $M^{\text{or}}_{(r,s)}(\Delta)$. We denote by $\text{ft} : M^{\text{or}}_{(r,s)}(\Delta, F) \to M_{(r,s)}(\Delta)$ the obvious *forgetful map* that disregards the information of the orientations.

Remark 2.16. Obviously, our constructions and results for non-oriented curves carry over immediately to the oriented case: $M^{\text{or}}_{(r,s)}(\Delta, F)$ is a polyhedral complex with cells $M^{\alpha}_{(r,s)}(\Delta, F)$ corresponding to the combinatorial types α of the oriented curves (which now include the data of the orientations of all edges). The forgetful map ft is a morphism of polyhedral complexes that is injective on each cell. There are evaluation maps on $M^{\text{or}}_{(r,s)}(\Delta, F)$ as in definition 2.8 (a) that are morphisms of polyhedral complexes; by abuse of notation we will write them as in the unoriented case as ev_F.

So far we have allowed any choice of orientations on the edges of our curves in $M_{(r,s)}^{\mathrm{or}}(\Delta, F)$. To ensure that the orientations are actually as explained in remark 2.14 we will now allow only certain types of vertices. In the rest of the paper we will study various kinds of oriented marked curves — broccoli curves in section 3, Welschinger curves in section 4, and bridge curves in section 5 — that differ mainly in their allowed vertex types. The following definition gives a complete list of all vertex types that will occur anywhere in this paper.

Definition 2.17 (Vertex types and multiplicities). We say that a vertex V of an oriented $(r+s)$-marked curve C is of a certain type if the number, parity (even or odd), and orientation of its adjacent edges is as in the following table. In addition, two arrows pointing in the same direction (as in the types (6b) and (8)) require these odd edges to be two unmarked ends with the same direction, and an arc (as in the types (6a) and (9)) means that these two odd edges must *not* be two unmarked ends with the same direction. Hence the type (6) splits up into the two subtypes (6a) and (6b). All other types in the list are mutually exclusive.

(1)	(2)	(3)	(4)
$m_V = 1$	$m_V = i^{a-1}$	$m_V = a \cdot i^{a-1}$	$m_V = a \cdot i^{a-1} = a \cdot i^{-1}$

(5)	(6)	(6a)	(6b)
$m_V = a \cdot i^{a-1}$	$m_V = i^{a-1}$	$m_V = i^{a-1}$	$m_V = i^{a-1} = i^{-1}$

(7)	(8)	(9)
$m_V = 1$	$m_V = -a$	$m_V = i^{a-1}$

In addition, each vertex V of one of the above types is assigned a *multiplicity* $m_V \in \mathbb{C}$ that can also be read off from the table. Here, the number a denotes the "complex vertex multiplicity" in the sense of Mikhalkin [Mik05], i.e. the absolute value of the determinant of two of the adjacent directions. For the type (8) it is the absolute value of the determinant of the two even adjacent directions.

If $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h)$ consists only of vertices of the above types, we denote by $n_\beta = n_\beta(C)$ the number of vertices in C of a given type β. In addition, we then define the *multiplicity* of C to be

$$m_C := \prod_{k=1}^{n} i^{\omega(y_k)-1} \cdot \prod_{V} m_V,$$

where the second product is taken over all vertices V of C. Although some of the vertex multiplicities are complex numbers, the following lemma shows that the curve multiplicity m_C is always real. In fact, the complex vertex multiplicities are just a computational trick that makes the "sign factor", i.e.

the power of i, the same for all the vertex types (2) to (6) (which will be the most important ones), leading to easier proofs in the rest of the paper.

Lemma 2.18. *Every oriented marked curve that has only vertices of the types in definition 2.17 has a real multiplicity.*

Proof. Let V be a vertex of C, and denote by $E_1, \ldots, E_q$ the adjacent unmarked edges (so $q \in \{2, 3, 4\}$ depending on the type of the vertex). Pick's theorem implies that the complex vertex multiplicity a as in definition 2.17 satisfies $a = \omega(E_1) + \cdots + \omega(E_q) \in \mathbb{Z}/2\mathbb{Z}$. By checking all vertex types we thus see that in each case

$$m_V \in \prod_{k=1}^{q} i^{\omega(E_k)-1} \cdot \mathbb{R}.$$

Now every unmarked edge is adjacent to exactly two vertices if it is bounded, and adjacent to exactly one vertex if it is unbounded. Hence

$$m_C \in \prod_{E} i^{2(\omega(E)-1)} \cdot \mathbb{R} = \mathbb{R},$$

where the sum is taken over all unmarked edges. $\qquad\square$

Example 2.19. The picture of example 2.4 and remark 2.14 shows an oriented marked curve C with $F(C) = \emptyset$. Its vertices V_1, V_2, V_3, labeled from left to right, are of the types (1), (3), and (6), respectively, so that e.g. $n_{(6)} = 1$. The vertex V_3 is also of type (6a). The multiplicities of the vertices are $m_{V_1} = 1$, $m_{V_2} = 2 \cdot i^{2-1} = 2i$, and $m_{V_3} = i^{2-1} = i$. As all unmarked ends of C have weight 1 the multiplicity of C is thus $m_C = -2$.

Let us now check that, with our list of allowed vertex types, in the situation of lemma 2.13 (b) the only way to orient a given curve is as explained in remark 2.14.

Lemma 2.20 (Uniqueness of the orientation of curves). *Let the notations and assumptions be as in lemma 2.13 (b). If there is a way to make C into an oriented curve with vertices of the types (1) to (7) and so that the orientations of the unmarked ends are as given by F, this must be the orientation that lets each unmarked edge point towards the unique unmarked and non-fixed end in the component of $\Gamma \backslash (x_1 \cup \cdots \cup x_{r+s})$ containing it.*

Proof. By lemma 2.13 (b) there is a unique orientation on C pointing on each unmarked edge towards the unmarked and non-fixed end in the component of $\Gamma \backslash (x_1 \cup \cdots \cup x_{r+s})$ containing the edge. Now assume that we have any orientation on C with vertices of types (1) to (7). Denote by Γ' the subgraph of Γ where these two orientations differ; we have to show that $\Gamma' = \emptyset$.

Note that Γ' is a bounded subgraph since the orientation on the ends is fixed by F. Moreover, Γ' cannot contain an edge adjacent to a marking since all possible vertex types (1), (5), (6), and (7) with markings require the orientation on the adjacent edges precisely as in remark 2.14. So if Γ' is non-empty it must have a 1-valent vertex somewhere that is not adjacent to a marking. This can only be a vertex of the types (2), (3), or (4), and the condition of Γ' being 1-valent means that the two orientations differ at exactly one adjacent edge. But this is impossible since both orientations have the property that they have one adjacent edge pointing outwards and two pointing inwards at this vertex. $\qquad\square$

We will end this section by computing the dimensions of the cells of $M^{\mathrm{or}}_{(r,s)}(\Delta, F)$.

Lemma 2.21. *Let $C \in M^{\mathrm{or}}_{(r,s)}(\Delta, F)$ be an oriented marked curve all of whose vertices are of the types listed in definition 2.17. Let α be the combinatorial type of C. Then the cell of $M^{\mathrm{or}}_{(r,s)}(\Delta, F)$ corresponding to α has dimension*

$$\dim \alpha = |\Delta| + r + n_{(7)} - n_{(8)} - 1 = 2(r+s) + |F| + n_{(9)}.$$

Proof. By remark 2.7 it suffices to show that the number of bounded edges of C is equal to

$$\text{both} \quad |\Delta| + r + n_{(7)}(C) - n_{(8)}(C) - 3 \quad \text{and} \quad 2(r+s) + |F| + n_{(9)}(C) - 2.$$

This is easily proven by induction on the number of vertices in C: if C has only one vertex (and thus no bounded edge) it has to be one of the types in definition 2.17, and the statement is easily checked in all of these cases. If the curve C has more than one vertex we cut it at any bounded edge into two parts C_1 and C_2, making the cut edge unbounded in both parts. If $C_i \in M_{(r_i,s_i)}(\Delta_i, F_i)$ for $i = 1,2$, then $r = r_1 + r_2$, $s = s_1 + s_2$, $|\Delta| = |\Delta_1| + |\Delta_2| - 2$, $|F| = |F_1| + |F_2| - 1$, and $n_\beta(C) = n_\beta(C_1) + n_\beta(C_2)$ for $\beta \in \{(7),(8),(9)\}$. The number of bounded edges of C is now just the number of bounded edges in C_1 and C_2 plus 1, i.e. by induction equal to

$$|\Delta_1| + r_1 + n_{(7)}(C_1) - n_{(8)}(C_1) - 3 + |\Delta_2| + r_2 + n_{(7)}(C_2) - n_{(8)}(C_2) - 3 + 1$$
$$= |\Delta| + r + n_{(7)}(C) - n_{(8)}(C) - 3$$

as well as

$$2(r_1 + s_1) + |F_1| + n_{(9)}(C_1) - 2 + 2(r_2 + s_2) + |F_2| + n_{(9)}(C_2) - 2 + 1$$
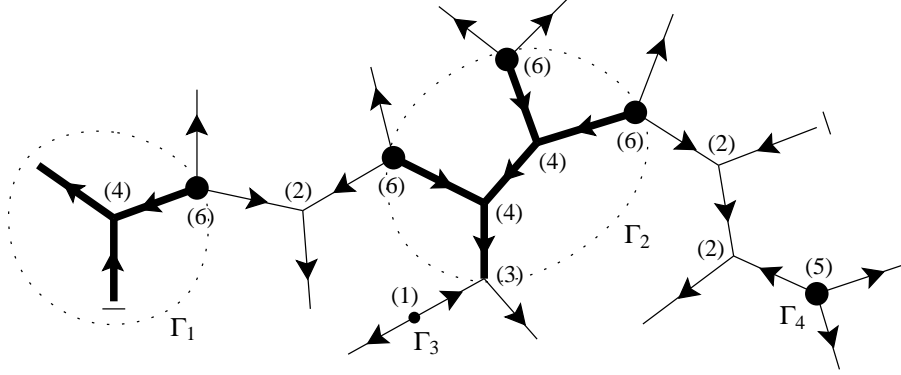$$= 2(r+s) + |F| + n_{(9)}(C) - 2.$$

$\square$

3. BROCCOLI CURVES

In this section we will introduce the most important type of curves considered in this paper: the broccoli curves. We define corresponding invariants, and show that they are independent of the chosen point conditions.

Broccoli curves can be defined with or without orientation. Both definitions have their advantages: the oriented one is easier to state and local at the vertices, whereas the unoriented one is easier to visualize (as one does not need to worry about orientations at all). So let us give both definitions and show that they agree for enumerative purposes.

Definition 3.1 (Broccoli curves). Let $r, s \geq 0$, let $\Delta = (v_1, \dots, v_n)$ be a collection of vectors in $\mathbb{Z}^2 \backslash \{0\}$, and let $F \subset \{1, \dots, n\}$.

(a) An oriented curve $C \in M^{\text{or}}_{(r,s)}(\Delta, F)$ all of whose vertices are of the types (1) to (6) of definition 2.17 is called an *oriented broccoli curve*.

(b) Let $C = (\Gamma, x_1, \dots, x_{r+s}, y_1, \dots, y_n, h) \in M_{(r,s)}(\Delta)$. Consider the subgraph Γ_{even} of Γ of all even edges (including the markings). The 1-valent vertices of Γ_{even} as well as the $y_i \subset \Gamma_{\text{even}}$ with $i \notin F$ are called the *stems* of Γ_{even}. We say that C is an *unoriented broccoli curve* (with set of fixed ends F) if

   (i) all complex markings are adjacent to 4-valent vertices;

   (ii) every connected component of Γ_{even} has exactly one stem.

Example 3.2. The picture below shows an oriented broccoli curve in which every allowed vertex type appears. We have labeled the vertices with their types. Note that by forgetting the orientations of the edges (and thus also disregarding the vertex types) one obtains an unoriented broccoli curve. Its subgraph Γ_{even} of even edges consists of all markings and thick edges. It has four connected components $\Gamma_1, \dots, \Gamma_4$, and each component has exactly one stem: the non-fixed unmarked end in Γ_1, the vertex of type (3) in Γ_2, and the unique vertices in Γ_3 and Γ_4.

Of course, to count these curves we have to fix the right number of conditions to get a finite answer. This dimension condition follows e.g. for oriented broccoli curves from lemma 2.21: we must have $r + 2s + |F| = |\Delta| - 1$ since $n_{(7)} = n_{(8)} = n_{(9)} = 0$.

Proposition 3.3 (Equivalence of oriented and unoriented broccoli curves). *Let $r, s \geq 0$, let $\Delta = (v_1, \ldots, v_n)$ be a collection of vectors in $\mathbb{Z}^2 \backslash \{0\}$, and let $F \subset \{1, \ldots, n\}$ such that $r + 2s + |F| = |\Delta| - 1$. Moreover, let $\mathscr{P} \in \mathbb{R}^{2(r+s)+|F|}$ be a collection of conditions in general position for $\mathrm{ev}_F : M_{(r,s)}(\Delta) \to \mathbb{R}^{2(r+s)+|F|}$ (see example 2.12).*

Then the forgetful map ft *of definition 2.15 gives a bijection between oriented and unoriented (r, s)-marked broccoli curves through $\mathscr{P}$ with degree Δ and set of fixed ends F.*

Proof. We have to prove three statements.

(a) ft maps oriented to unoriented broccoli curves through $\mathscr{P}$: Let $C \in M^{\mathrm{or}}_{(r,s)}(\Delta, F)$ be an oriented broccoli curve. The list of allowed vertex types for C implies immediately that C then satisfies condition (i) of definition 3.1.
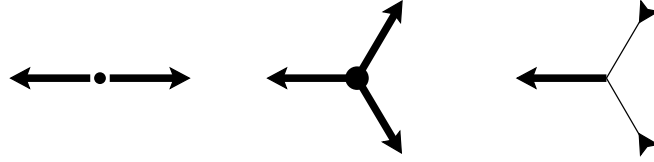
To show (ii) let Γ' be a connected component of Γ_{even}. If Γ' contains no vertex of type (4) it can only be a single marking (types (1) or (5)) or a single unmarked edge with possibly attached markings (vertex types (3) together with (6), (3) with a fixed unmarked end, or (6) with a non-fixed unmarked end), an in each of these cases condition (ii) is satisfied. If there are vertices of type (4) they must form a tree in Γ', and obviously every such tree made up from type (4) vertices has exactly one outgoing end. This unique outgoing end must be a non-fixed end of C or connected to a type (3) vertex, hence in any case it leads to a stem. On the other hand, the incoming ends of the tree must be fixed ends of C or connected to a type (6) vertex, i.e. they never lead to a stem. Consequently, Γ' satisfies condition (ii).

(b) ft is injective on the set of curves through $\mathscr{P}$: Note that the conditions of lemma 2.13 (b) are satisfied by the dimension condition of lemma 2.21 and our list of allowed vertex types. Hence lemma 2.20 implies that there is at most one possible orientation on C.

(c) ft is surjective on the set of curves through $\mathscr{P}$: Let $C \in M_{(r,s)}(\Delta)$ be an unoriented broccoli curve through $\mathscr{P}$ with set of fixed ends F. Then by (i) the curve C has s 4-valent vertices at the complex markings, so by [GM08] proposition 2.11 the combinatorial type of C has dimension $|\Delta| - 1 + r - \sum_V (\mathrm{val}\, V - 3) = 2(r+s) + |F| - \sum_V (\mathrm{val}\, V - 3)$, with the sum taken over all vertices V that are not adjacent to a complex marking. But as $\mathscr{P}$ is in general position this dimension cannot be less than $2(r+s) + |F|$. So we see that all vertices without adjacent complex marking are 3-valent, and that the combinatorial type of C has dimension equal to $2(r+s) + |F|$. Hence we can apply lemma 2.13 (b) again to conclude that there is an orientation on C that points on each edge towards the unique non-fixed unmarked end in $\Gamma \backslash (x_1 \cup \cdots \cup x_{r+s})$.

It remains to be shown that with this orientation the only vertex types occurring in C are (1) to (6). For this, note that for a vertex V

- as we have said above, V is 4-valent if there is a complex marking at V, and 3-valent otherwise;

- by the construction of the orientation, all edges at V are oriented outwards if there is a marking at V, and exactly one edge is oriented outwards otherwise;

- by the balancing condition, it is impossible that exactly one edge at V is odd.

With these restrictions, the only possible vertex types besides (1) to (6) would be the ones in the picture below.

To exclude these three cases, note that in all of them V would be contained in a connected component Γ' of Γ_{even} that contains at least one unmarked edge. So let us consider such a component, and let $W \in \Gamma' \cap (\Gamma \backslash \Gamma')$ be a vertex where Γ' meets the complement of Γ'. Then there must be an odd as well as an unmarked even edge in Γ at W, so by the balancing condition as above there are exactly two odd edges and one even unmarked edge at W. Hence W is a stem if and only if there is no marking at W. So a connection in $\Gamma \backslash (x_1 \cup \cdots \cup x_{r+s})$ from a point in the interior of Γ' to a non-fixed unmarked end can only be via a stem — which is unique by (ii). This means that every point in the interior of Γ' must be connected in $\Gamma \backslash (x_1 \cup \cdots \cup x_{r+s})$ to the stem. In particular, the interior of Γ' can have no further markings, which rules out the first two vertex types in the picture above. The third vertex type is impossible since this would have to be the stem and thus the connection from Γ' to the non-fixed unmarked end, which does not match with the orientation of the even edge.

□

Let us now make the obvious definition of the enumerative invariants corresponding to broccoli curves. Proposition 3.3 tells us that it does not matter whether we count oriented or unoriented broccoli curves. We choose the oriented ones here as their definition is easier. So we make the convention that from now on *a broccoli curve will always mean an oriented broccoli curve*.

Notation 3.4. We denote by $M^B_{(r,s)}(\Delta, F)$ the closure of the space of all broccoli curves in $M^{\text{or}}_{(r,s)}(\Delta, F)$; this is obviously a polyhedral subcomplex. By lemma 2.21 it is non-empty only if the dimension condition $r + 2s + |F| = |\Delta| - 1$ is satisfied. Moreover, in this case it is of pure dimension $2(r+s) + |F|$, and its maximal open cells correspond exactly to the broccoli curves in $M^B_{(r,s)}(\Delta, F)$.

Definition 3.5 (Broccoli invariants)**.** As above, let $r, s \geq 0$, let $\Delta = (v_1, \ldots, v_n)$ be a collection of vectors in $\mathbb{Z}^2 \backslash \{0\}$, and let $F \subset \{1, \ldots, n\}$ such that $r + 2s + |F| = |\Delta| - 1$. Moreover, let $\mathscr{P} \in \mathbb{R}^{2(r+s)+|F|}$ be a collection of conditions in general position for broccoli curves, i.e. for the evaluation map $\text{ev}_F : M^B_{(r,s)}(\Delta, F) \to \mathbb{R}^{2(r+s)+|F|}$. Then we define the *broccoli invariant*

$$N^B_{(r,s)}(\Delta, F, \mathscr{P}) := \frac{1}{|G(\Delta, F)|} \cdot \sum_C m_C,$$

where the sum is taken over all broccoli curves C in with degree Δ, set of fixed ends F, and $\text{ev}(C) = \mathscr{P}$. The group $G(\Delta, F)$ as in definition 2.8 (b) takes care of the overcounting of curves due to relabeling the non-fixed unmarked ends. The sum is finite by the dimension statement of notation 3.4, and the multiplicity m_C is as in definition 2.17.
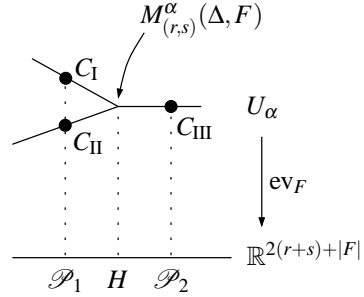
The main result of this section — and in fact the most important point that distinguishes our new invariants from the otherwise quite similar Welschinger invariants that we will study in section 4 — is that broccoli invariants are always independent of the choice of conditions $\mathscr{P}$.

Theorem 3.6. *The broccoli invariants $N^B_{(r,s)}(\Delta, F, \mathscr{P})$ are independent of the collection of conditions $\mathscr{P}$. We will thus usually write them simply as $N^B_{(r,s)}(\Delta, F)$ (or $N^B_{(r,s)}(\Delta)$ for $F = \emptyset$).*

Proof. The proof follows from a local study of the moduli space $M^B_{(r,s)}(\Delta, F)$. Compared to the one for ordinary tropical curves in [GM07b] theorem 4.8 it is very similar in style and conceptually not more complicated; there are just (many) more cases to consider because we have to distinguish orientations as well as even and odd edges.

By definition, the multiplicity of a curve depends only on its combinatorial type. So it is obvious that the function $\mathscr{P} \mapsto N^B_{(r,s)}(\Delta, F, \mathscr{P})$ is *locally* constant on the open subset of $\mathbb{R}^{2(r+s)+|F|}$ of conditions in general position for broccoli curves, and may jump only at the image under ev_F of the boundary of top-dimensional cells of $M^B_{(r,s)}(\Delta, F)$. This image is a union of polyhedra in $\mathbb{R}^{2(r+s)+|F|}$ of positive codimension. It suffices to show that the function $\mathscr{P} \mapsto N^B_{(r,s)}(\Delta, F, \mathscr{P})$ is locally constant around a cell in this image of codimension 1 in $\mathbb{R}^{2(r+s)+|F|}$ since any two top-dimensional cells of $\mathbb{R}^{2(r+s)+|F|}$ can be connected to each other through codimension-1 cells.

So let α be a combinatorial type in $M^B_{(r,s)}(\Delta, F)$ of dimension $2(r+s) + |F| - 1$ such that ev_F is injective on $M^\alpha_{(r,s)}(\Delta, F)$ and thus maps this cell to a unique hyperplane H in $\mathbb{R}^{2(r+s)+|F|}$. As in the picture on the right let $U_\alpha \subset M^B_{(r,s)}(\Delta, F)$ be the open subset consisting of $M^\alpha_{(r,s)}(\Delta, F)$ together with all adjacent top-dimensional cells of $M^B_{(r,s)}(\Delta, F)$. To prove the theorem we will show that for a point $\mathscr{P}$ in a neighborhood of $\mathrm{ev}_F(M^\alpha_{(r,s)}(\Delta, F))$ the sum of the multiplicities of the curves in $U_\alpha \cap \mathrm{ev}_F^{-1}(\mathscr{P})$ does not depend on $\mathscr{P}$, i.e. is the same on both sides of H. In our picture this would just mean that $m_{\mathrm{I}} + m_{\mathrm{II}} = m_{\mathrm{III}}$, where $m_{\mathrm{I}}, m_{\mathrm{II}}, m_{\mathrm{III}}$ denote the multiplicities of $C_{\mathrm{I}}, C_{\mathrm{II}}, C_{\mathrm{III}}$, respectively.
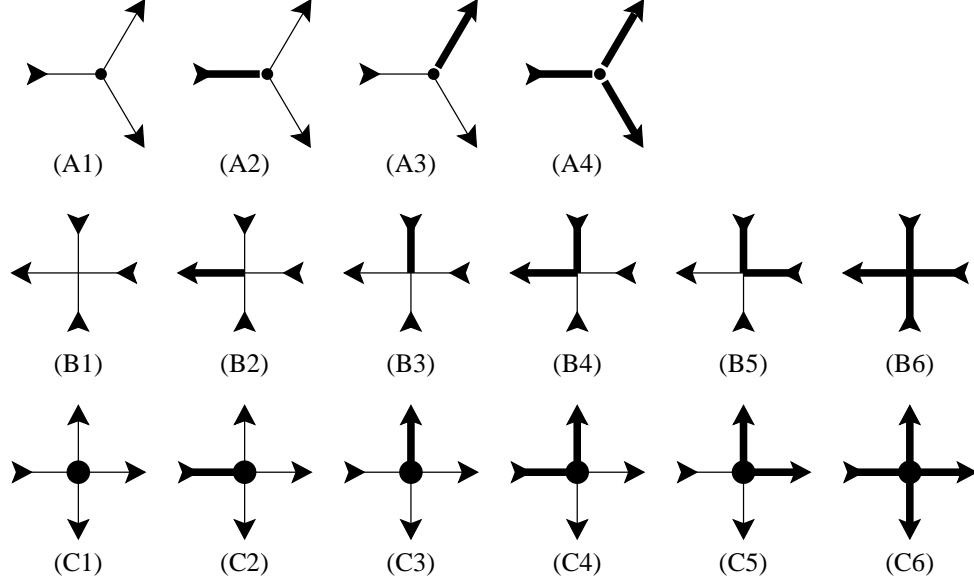

Actually, we will show this in a slightly different form: to each codimension-0 type α_k in U_α we will associate a so-called *H-sign* σ_k that is 1 or -1 depending on the side of H on which $\mathrm{ev}_F(M^{\alpha_k}_{(r,s)}(\Delta, F))$ lies (it will be 0 if $\mathrm{ev}_F(M^{\alpha_k}_{(r,s)}(\Delta, F)) \subset H$). So in the picture above on the right we could take $\sigma_{\mathrm{I}} = \sigma_{\mathrm{II}} = 1$ and $\sigma_{\mathrm{III}} = -1$. We then obviously have to show that $\sum_k \sigma_k m_k = 0$, where the sum is taken over all top-dimensional cells adjacent to α.

To prove this, we will start by listing all codimension-1 combinatorial types α in $M^B_{(r,s)}(\Delta, F)$. They are obtained by shrinking the length of a bounded edge in a broccoli curve to zero, thereby merging two vertices into one. Depending on the merging vertex types we distinguish the following cases:

(A) a vertex (1) merging with a vertex (2)/(3), leading to a 4-valent vertex with one real marking, two outgoing edges, and one incoming edge.

(B) a vertex (2)/(3)/(4) merging with a vertex (2)/(3)/(4), leading to a 4-valent vertex with no marking, one outgoing edge, and three incoming edges.

(C) a vertex (5)/(6) merging with a vertex (2)/(3)/(4), leading to a 5-valent vertex with one complex marking, three outgoing edges, and one incoming edge.

More precisely, noting that by the balancing condition it is impossible to have exactly one odd edge at a vertex, the cases (A), (B), and (C) split up into the following possibilities depending on the orientation and parity of the adjacent edges.

(A1)   (A2)   (A3)   (A4)

(B1)   (B2)   (B3)   (B4)   (B5)   (B6)

(C1)   (C2)   (C3)   (C4)   (C5)   (C6)

Next, we will list the adjacent codimension-0 types in $M^B_{(r,s)}(\Delta, F)$ (called *resolutions*) that make up U_α in the cases (A), (B), and (C). In this picture, the dashed lines can be even or odd depending on which of the subcases (A·), (B·), (C·) we are in. The vectors $v_1, \ldots, v_4$ will be used in the computations below; they are always meant to be oriented outwards (i.e. *not* necessarily in the direction of the orientation of the edge), so that $v_1 + v_2 + v_3 = 0$ in case (A) and $v_1 + v_2 + v_3 + v_4 = 0$ in the cases (B) and (C).

(A)   I   II

(B)   I   II   III

(C)   I   II   III

Note that the allowed vertex types for broccoli curves fix the orientation of the newly inserted bounded edge in all these resolutions; it is already indicated in the picture above. Moreover, the requirement that there cannot be exactly one odd edge at a vertex fixes the parity of the new bounded edge in all cases except (B1) and (C1). In the (B1) and (C1) cases, there are two possibilities: the four vectors $v_1, \ldots, v_4$ can either be all the same in $(\mathbb{Z}_2)^2$ (in which case the new bounded edge joining V and W is even in all three types I, II, III; we call this case (B1$_3$) and (C1$_3$), respectively), or they make up all three non-zero equivalence classes in $(\mathbb{Z}_2)^2$ (in which case the new bounded edge is even in exactly one of the types I, II, III; we call this case (B1$_1$) and (C1$_1$), respectively). In the (B1$_1$) and (C1$_1$) cases, we can assume by symmetry that the even bounded edge occurs in type I. So in total we now have 18 codimension-1 cases (A1), $\ldots$, (A4), (B1$_1$), (B1$_3$), (B2),$\ldots$(B6), (C1$_1$), (C1$_3$), (C2),$\ldots$(C6) to consider, and in each of these cases we know the resolutions together with all parities and orientations of all edges of the curves — in particular, with the vertex types of V and W (as in the picture above). For example, in case (B6) the new bounded edge must be even in all three resolutions. Hence in all three resolutions all edges are even, and thus both vertices V and W are of type (4).

The following table lists the vertex types for V and W for all resolutions I, II, III of all codimension-1 cases. The symbol "—" means that the required vertex type is not allowed in broccoli curves and thus that a corresponding codimension-0 cell does not exist. The columns labeled m_* and μ_*/μ_* will be explained below.

codim-1 case	resolution I			resolution II			
	V	W	m_{I}	V	W	$\mu_{\mathrm{II}}/\mu_{\mathrm{I}}$	m_{II}
A1	(2)	(1)	1	(2)	(1)	-1	1
A2	(3)	(1)	(v_1,v_2)	(3)	(1)	1	(v_1,v_3)
A3	—	(1)	0	(3)	—	1	0
A4	(4)	—	0	(4)	—	1	0

codim-1 case	resolution I			resolution II				resolution III			
	V	W	m_{I}	V	W	$\mu_{\mathrm{II}}/\mu_{\mathrm{I}}$	m_{II}	V	W	$\mu_{\mathrm{III}}/\mu_{\mathrm{I}}$	m_{III}
B1$_1$	(3)	—	0	(2)	(2)	1	1	(2)	(2)	-1	1
B1$_3$	(3)	—	0	(3)	—	1	0	(3)	—	1	0
B2	—	(2)	0	—	(2)	1	0	—	(2)	1	0
B3	(3)	(2)	(v_1,v_2)	(2)	(3)	1	(v_4,v_2)	(2)	(3)	-1	(v_2,v_3)
B4	(4)	—	0	—	(3)	1	0	—	(3)	1	0
B5	(3)	(3)	$(v_1,v_2)(v_3,v_4)$	(3)	(3)	1	$(v_1,v_3)(v_4,v_2)$	(3)	(4)	1	$(v_1,v_4)(v_2,v_3)$
B6	(4)	(4)	$(v_1,v_2)(v_3,v_4)$	(4)	(4)	1	$(v_1,v_3)(v_4,v_2)$	(4)	(4)	1	$(v_1,v_4)(v_2,v_3)$

codim-1 case	resolution I			resolution II				resolution III			
	V	W	m_{I}	V	W	$\mu_{\mathrm{II}}/\mu_{\mathrm{I}}$	m_{II}	V	W	$\mu_{\mathrm{III}}/\mu_{\mathrm{I}}$	m_{III}
C1$_1$	(3)	(6)	(v_1,v_2)	(2)	(5)	1	(v_4,v_2)	(2)	(5)	-1	(v_2,v_3)
C1$_3$	(3)	(6)	(v_1,v_2)	(3)	(6)	1	(v_1,v_3)	(3)	(6)	1	(v_1,v_4)
C2	(3)	(5)	$(v_1,v_2)(v_3,v_4)$	(3)	(5)	1	$(v_1,v_3)(v_4,v_2)$	(3)	(5)	1	$(v_1,v_4)(v_2,v_3)$
C3	—	(5)	0	(2)	(6)	1	1	(2)	(6)	-1	1
C4	(4)	(6)	(v_1,v_2)	(3)	(6)	1	(v_1,v_3)	(3)	(6)	1	(v_1,v_4)
C5	—	(6)	0	—	(6)	1	0	(3)	—	1	0
C6	(4)	—	0	(4)	—	1	0	(4)	—	1	0

Let us now determine the H-sign of the resolutions above, i.e. figure out which of them occur on which side of H. To do this we set up the system of linear equations determining the lengths of the bounded edges of the curve in terms of the positions of the markings in $\mathbb{R}^2$. For such a given position

of the markings (on the one or on the other side of H), a given resolution type is then possible if and only if the required length of the new bounded edge is positive.

More concretely, let a be the length of the newly created bounded edge, and denote by $P \in \mathbb{R}^2$ in the cases (A) and (C) the required image point for the marking. In the cases (A) and (C) the end v_1 is fixed, so say there is another marking on the v_1 end at a distance of l_1 on the graph that is required to map to a point $P_1 \in \mathbb{R}^2$. In the case (B) the ends v_2, v_3, and v_4 are fixed, so we do the same then with lengths l_2, l_3, l_4 and points $P_2, P_3, P_4 \in \mathbb{R}^2$. As an example, these notions are illustrated for the resolution I in the following picture.

(A)-I   (B)-I   (C)-I

The systems of linear equations that determine the relative positions of $P, P_1, \ldots, P_4$ in terms of $a, l_1, \ldots, l_4$ are then as follows (where all entries are in $\mathbb{R}^2$ and thus each row stands for two equations).

(A)-I

l_1	a	
$-v_1$	v_3	$P - P_1$

(A)-II

l_1	a	
$-v_1$	v_2	$P - P_1$

(B)-I

l_2	l_3	l_4	a	
$-v_2$	v_3	0	$v_3 + v_4$	$P_3 - P_2$
$-v_2$	0	v_4	$v_3 + v_4$	$P_4 - P_2$

(B)-II

l_2	l_3	l_4	a	
$-v_2$	v_3	0	$v_1 + v_3$	$P_3 - P_2$
$-v_2$	0	v_4	0	$P_4 - P_2$

(B)-III

l_2	l_3	l_4	a	
$-v_2$	v_3	0	0	$P_3 - P_2$
$-v_2$	0	v_4	$v_1 + v_4$	$P_4 - P_2$

(C)-I

l_1	a	
$-v_1$	$v_3 + v_4$	$P - P_1$

(C)-II

l_1	a	
$-v_1$	$v_2 + v_4$	$P - P_1$

(C)-III

l_1	a	
$-v_1$	$v_2 + v_3$	$P - P_1$

To determine a in terms of $P, P_1, \ldots, P_4$ we use Cramer's rule: if M is the (quadratic) matrix of a system of linear equations as above and M' the matrix obtained from M by replacing the a-column by the right hand side of the equation, then $a = \det M' / \det M$. But within a case (A), (B), (C) the matrix M' does not depend on the resolution I, II, III, and thus it is simply the sign of $\det M$ that tells us whether a is positive or negative, i.e. whether this resolution exists for the chosen points $P, P_1, \ldots, P_4$. We can therefore take the H-sign to be the sign of $\det M$ (note that this will be 0 if and only if the relative position of $P, P_1, \ldots, P_4$ is not determined uniquely by the equations and thus if and only if the codimension-0 cell maps to H). An elementary computation of the determinants shows that these H-signs are as in the following table, where (v_i, v_j) stands for the determinant of the 2×2 matrix with columns v_i, v_j (and where we have used $v_1 + v_2 + v_3 = 0$ in case (A) as well as $v_1 + v_2 + v_3 + v_4 = 0$ in the cases (B) and (C)).

	H-sign for I	H-sign for II	H-sign for III
(A)	$\mathrm{sign}(v_1, v_2)$	$\mathrm{sign}(v_1, v_3)$	
(B)	$\mathrm{sign}\big((v_1, v_2)(v_3, v_4)\big)$	$\mathrm{sign}\big((v_1, v_3)(v_4, v_2)\big)$	$\mathrm{sign}\big((v_1, v_4)(v_2, v_3)\big)$
(C)	$\mathrm{sign}(v_1, v_2)$	$\mathrm{sign}(v_1, v_3)$	$\mathrm{sign}(v_1, v_4)$

Note that these H-signs follow a special pattern: for each of the vertices V and W that is of type (2), (3), or (4) we get a factor of $\mathrm{sign}(v_i, v_j)$ in the H-sign of the resolution, where $(i, j) \in \{(1,2),(1,3),(1,4),(3,4),(4,2),(2,3)\}$ is the unique pair such that the v_i and v_j edges are adjacent to the vertex. On the other hand, by definition 2.17 the multiplicity of such a vertex is 1 in type (1), $i^{|(v_i,v_j)|-1}$ in types (2) and (6), and $|(v_i, v_j)| \cdot i^{|(v_i,v_j)|-1}$ in types (3), (4), and (5). If one replaces $|(v_i, v_j)|$ by $-|(v_i, v_j)|$ in these expressions, the vertex multiplicities remain the same for the types (1), (5) and (6), and are replaced by their negatives for the types (2), (3), and (4). It follows that the H-sign can be taken care of by replacing $a = |(v_i, v_j)|$ by (v_i, v_j) in the vertex multiplicities of definition 2.17 for V and W.

More precisely, if σ denotes the H-sign and m the multiplicity of a curve in a given resolution, then $\sigma m = \lambda \, \tilde{m}_V \tilde{m}_W$, where $\tilde{m}_V$ and $\tilde{m}_W$ are the multiplicities of the vertices V and W as in definition 2.17 with a replaced by (v_i, v_j) as above, and λ is the product of the vertex multiplicities of all other vertices. To show that the sum of these numbers over all resolutions is zero we can obviously divide by the constant λ (which is the same for the resolutions I, II, III) and only consider $\tilde{m}_V \tilde{m}_W$. Let us split this number as $\tilde{m}_V \tilde{m}_W = \mu \, m$, where μ collects all factors $i^{(v_i,v_j)-1}$ and m all factors (v_i, v_j) for V and W. The values for $m = m_{\mathrm{I}}, m_{\mathrm{II}}, m_{\mathrm{III}}$ are listed in the table of resolutions above. As for μ, note that this number is

- in case (A): $\mu_{\mathrm{I}} := i^{(v_1,v_2)-1}$ for I and $\mu_{\mathrm{II}} := i^{(v_1,v_3)-1}$ for II;
- in cases (B) and (C): $\mu_{\mathrm{I}} := i^{(v_1,v_2)+(v_3,v_4)-2}$ for I, $\mu_{\mathrm{II}} := i^{(v_1,v_3)+(v_4,v_2)-2}$ for II, and $\mu_{\mathrm{III}} := i^{(v_1,v_4)+(v_2,v_3)-2}$ for III.

To simplify these expressions we divide them by μ_{I} and get (using $v_1 + v_2 + v_3 = 0$ in (A) and $v_1 + v_2 + v_3 + v_4 = 0$ in (B) and (C))

- in case (A): $\mu_{\mathrm{II}}/\mu_{\mathrm{I}} = i^{2(v_2,v_1)} = (-1)^{(v_2,v_1)}$;
- in cases (B) and (C): $\mu_{\mathrm{II}}/\mu_{\mathrm{I}} = i^{2(v_2,v_1)} = (-1)^{(v_2,v_1)}$ and $\mu_{\mathrm{III}}/\mu_{\mathrm{I}} = i^{2(v_1,v_4)} = (-1)^{(v_1,v_4)}$.

The values for these quotients are also listed in the table of resolutions above. Using these values for the quotients and $m_{\mathrm{I}}, m_{\mathrm{II}}, m_{\mathrm{III}}$, one can now easily check the required statement

$$\mu_{\mathrm{I}} \cdot m_{\mathrm{I}} + \mu_{\mathrm{II}} \cdot m_{\mathrm{II}} + \mu_{\mathrm{III}} \cdot m_{\mathrm{III}} = \mu_{\mathrm{I}} \cdot \left(m_{\mathrm{I}} + \mu_{\mathrm{II}}/\mu_{\mathrm{I}} \cdot m_{\mathrm{II}} + \mu_{\mathrm{III}}/\mu_{\mathrm{I}} \cdot m_{\mathrm{III}} \right) = 0$$

in all 18 codimension-1 cases, using the identities

- $(v_1, v_2) + (v_1, v_3) = 0$ for (A),
- $(v_1, v_2) + (v_4, v_2) + (v_3, v_2) = 0$, $(v_1, v_2)(v_3, v_4) + (v_1, v_3)(v_4, v_2) + (v_1, v_4)(v_2, v_3) = 0$, and $(v_1, v_2) + (v_1, v_3) + (v_1, v_4) = 0$ for (B) and (C),

that follow from $v_1 + v_2 + v_3 = 0$ and $v_1 + v_2 + v_3 + v_4 = 0$, respectively. $\qquad \square$

4. WELSCHINGER CURVES

In this section we define tropical curves that we call Welschinger curves. Their count (for certain choices of Δ) yields Welschinger invariants, i.e. numbers of real rational curves on a toric Del Pezzo surface Σ belonging to an ample linear system D and passing through a generic conjugation invariant set of $-K_\Sigma \cdot D - 1$ points, weighted with ± 1, depending on the nodes of the curve.

As we have mentioned already in the introduction, we will parametrize even non-fixed unmarked ends of Welschinger curves as two ends of half the weight — this way we can avoid this kind of splitting on the bridges of section 5. We will refer to such ends, i.e. pairs of non-fixed ends of the same odd direction adjacent to the same 4-valent vertex, as double ends. In the following, we will first settle how to deal with these double ends. Then we define oriented and unoriented Welschinger curves and prove that they are equivalent. We relate unoriented Welschinger curves

to tropical curves in other literature that are counted to determine Welschinger invariants, cite the Correspondence Theorem, and discuss some invariance and non-invariance properties of tropical Welschinger numbers.
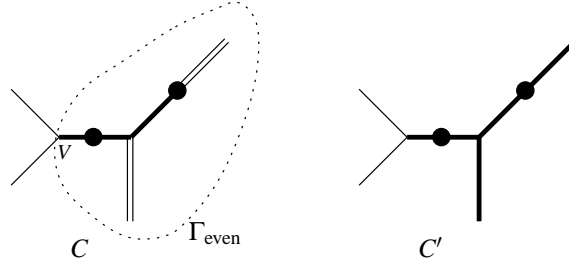
Definition 4.1 (Double ends and end-gluing). Let α be a combinatorial type of $M_{(r,s)}(\Delta)$ with $\Delta = (v_1, \ldots, v_n)$, and let $F \subset \{1, \ldots, n\}$ be a set of fixed ends. Assume that there are exactly k pairs $i_1 < j_1, \ldots, i_k < j_k$ in $\{1, \ldots, n\} \setminus F$ such that the unmarked ends y_{i_l} and y_{j_l} have the same odd direction and are adjacent to the same 4-valent vertex, for all $l = 1, \ldots, k$. We refer in the following to such a pair of ends as a *double end*. We then set

$$\Delta' = \big((v(y_i) : i \neq i_1, j_1, \ldots, i_k, j_k), (2 \cdot v(y_{i_1}), \ldots, 2 \cdot v(y_{i_k}))\big).$$

Moreover, we define α' by gluing each pair of double ends y_{i_l} and y_{j_l} to one unmarked end of direction $2 \cdot v(y_{i_l})$, and denote by $F' \subset \{1, \ldots, n-k\}$ the set of entries of Δ' corresponding to the fixed ends F in Δ. There is then an associated map $M_{(r,s)}^{\alpha}(\Delta) \to M_{(r,s)}^{\alpha'}(\Delta')$ which we call the *end-gluing map*.

The analogous end-gluing map $(M_{(r,s)}^{\mathrm{or}})^{\alpha}(\Delta, F) \to (M_{(r,s)}^{\mathrm{or}})^{\alpha'}(\Delta', F')$ also exists for oriented curves. The map sending a combinatorial type α of $M_{(r,s)}(\Delta)$ as above to α' is injective, because if we want to produce a preimage α from α', we just have to split the last k ends of Δ', producing 4-valent vertices.

Example 4.2. The following picture shows a curve C and its image C' under the end-gluing map. Although mainly following convention 2.3, we draw double ends separately even though this is actually a feature of the graph Γ and cannot be seen in $h(\Gamma)$.

$$C \qquad \Gamma_{\mathrm{even}} \qquad\qquad C'$$

Remark 4.3. It follows from example 2.12 that if a collection of conditions $\mathscr{P} \in \mathbb{R}^{2(r+s)+|F|}$ as in remark 2.9 is in general position for $\mathrm{ev}_F : M_{(r,s)}^{\alpha}(\Delta) \to \mathbb{R}^{2(r+s)+|F|}$ then it is also in general position after end-gluing for $\mathrm{ev}_{F'} : M_{(r,s)}^{\alpha'}(\Delta') \to \mathbb{R}^{2(r+s)+|F|}$, and vice versa. Notice also that $\dim M_{(r,s)}^{\alpha}(\Delta) = \dim M_{(r,s)}^{\alpha'}(\Delta')$: by [GM08] proposition 2.11 a combinatorial type has dimension $|\Delta| - 1 + r + s - \sum_V(\mathrm{val}(V) - 3)$ where the sum goes over all vertices V of Γ, and the end-gluing map decreases the number of entries of Δ by the same number as it decreases the number of 4-valent vertices. As orienting the edges does not change dimensions we conclude that the end-gluing map does not change the dimension of combinatorial types of oriented curves either.
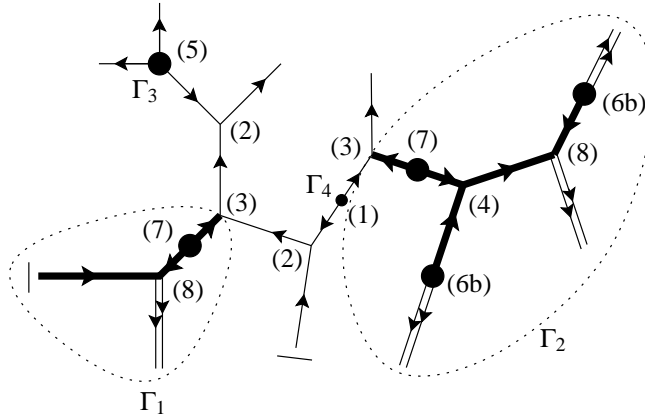
Definition 4.4 (Γ_{even} and roots). Let $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h) \in M_{(r,s)}(\Delta)$. Let C' be the image of C under the end-gluing map of definition 4.1 and call the underlying graph Γ'. Consider the subgraph Γ'_{even} of Γ' of all even edges (including the markings), and its preimage Γ_{even}. That is, Γ_{even} consists of all even edges and all double ends of Γ. Vertices of $\Gamma_{\mathrm{even}} \cap \overline{\Gamma \setminus \Gamma_{\mathrm{even}}}$ as well as unmarked non-fixed *even* ends of Γ_{even} are called the *roots* of Γ_{even}.

Example 4.5. For the curve of example 4.2, the part Γ_{even} is encircled. It has one root, namely the vertex denoted by V.

Definition 4.6 (Welschinger curves). Let $r, s \geq 0$, let $\Delta = (v_1, \ldots, v_n)$ be a collection of vectors in $\mathbb{Z}^2 \setminus \{0\}$, and let $F \subset \{1, \ldots, n\}$.

(a) An oriented curve $C \in M^{\mathrm{or}}_{(r,s)}(\Delta, F)$ all of whose vertices are of the types (1) to (5), (6b), (7), or (8) of definition 2.17 is called an *oriented Welschinger curve*.

(b) Let $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h) \in M_{(r,s)}(\Delta)$, and let Γ_{even} be as in definition 4.4. We say that C is an *unoriented Welschinger curve* (with set of fixed ends F) if

   (i) complex markings are adjacent to 4-valent vertices, or non-isolated in Γ_{even};

   (ii) each connected component of Γ_{even} has a unique root.

Example 4.7. The following picture shows an oriented Welschinger curve with an even and an odd fixed end. As in example 4.2, we indicate double ends in the picture while otherwise following convention 2.3. Each vertex is labeled with its type, every allowed vertex type occurs. If we forget the orientations of the edges, we get an unoriented Welschinger curve. There are four connected components of Γ_{even}. Γ_3 consists of a complex marking and Γ_4 of a real marking. Γ_1 and Γ_2 both have one root, namely the vertex of type (3). Three complex markings are adjacent to 4-valent vertices, four are non-isolated in Γ_{even}.

As for broccoli curves, we want to show that oriented and unoriented Welschinger curves are equivalent for enumerative purposes. The following remark and lemma are needed as preparation.

Remark 4.8. Let $C \in M^{\mathrm{or}}_{(r,s)}(\Delta, F)$ be an oriented Welschinger curve.

   (a) By lemma 2.21, the curve C has $|\Delta| - |F| = r + 2s + 1 - n_{(7)} + n_{(8)}$ outward pointing ends. In particular, if $|\Delta| - 1 = r + 2s + |F|$ then $n_{(7)} = n_{(8)}$.

   (b) If C consists only of vertices of types (4), (6b), (7) and (8), then we have $r = 0$, $s = n_{(6b)} + n_{(7)}$, and the number of odd outward pointing ends is $2n_{(6b)} + 2n_{(8)}$. Hence in this case it follows from (a) that C has exactly $1 + n_{(7)} - n_{(8)}$ even outward pointing ends.

Lemma 4.9. *Let $|\Delta| - 1 = r + 2s + |F|$, let $C \in M^{\mathrm{or}}_{(r,s)}(\Delta, F)$ be an oriented Welschinger curve, and let Γ_{even} be as in definition 4.4. Then every connected component of Γ_{even} has exactly one root.*

Proof. If $\Gamma_{\mathrm{even}} = \Gamma$ then Γ has only vertices of type (4), (6b), (7), and (8). By remark 4.8 (a) we have $n_{(7)} = n_{(8)}$, so from remark 4.8 (b) it then follows that Γ has exactly one even outward pointing end, which is the unique root.

If $\Gamma_{\mathrm{even}} \neq \Gamma$, every connected component $\tilde{\Gamma}$ of Γ_{even} needs to be adjacent to odd edges which are not double ends. The only allowed vertex type for oriented Welschinger curves to which both even edges (resp. double edges) and odd edges (which are not double ends) are adjacent is type (3). Each vertex of type (3) yields a 1-valent vertex in Γ_{even}. Remove these 1-valent vertices from the component $\tilde{\Gamma}$, and call the resulting graph $\tilde{\Gamma}^\circ$. A vertex of type (3) leads to an outward pointing end of $\tilde{\Gamma}^\circ$. Note

that $\tilde{\Gamma}^{\circ}$ has vertices of types (4), (6b), (7), and (8). Thus by remark 4.8 (b) we have $n_{(8)}^{\tilde{\Gamma}^{\circ}} \leq n_{(7)}^{\tilde{\Gamma}^{\circ}}$, where the superscripts indicate that we refer to numbers of vertices of $\tilde{\Gamma}^{\circ}$. By remark 4.8 (a) we have $n_{(7)}^{C} = n_{(8)}^{C}$. Since any vertex of type (7) or (8) belongs to exactly one graph $\tilde{\Gamma}^{\circ}$ associated to a connected component $\tilde{\Gamma}$ of Γ_{even}, and since the inequality $n_{(8)}^{\tilde{\Gamma}^{\circ}} \leq n_{(7)}^{\tilde{\Gamma}^{\circ}}$ holds for any such $\tilde{\Gamma}$, we conclude that it is an equality. Then by remark 4.8 (b) every $\tilde{\Gamma}^{\circ}$ has exactly one even outward pointing end. It follows that every $\tilde{\Gamma}$ has exactly one root. $\qquad\square$

With this preparation we can prove the following statement analogously to proposition 3.3.

Proposition 4.10 (Equivalence of oriented and unoriented Welschinger curves)**.** *Let $r, s \geq 0$, let $\Delta = (v_1, \ldots, v_n)$ be a collection of vectors in $\mathbb{Z}^2 \backslash \{0\}$, and let $F \subset \{1, \ldots, n\}$ such that $r + 2s + |F| = |\Delta| - 1$. Moreover, let $\mathscr{P} \in \mathbb{R}^{2(r+s)+|F|}$ be a collection of conditions in general position for $\text{ev}_F : M_{(r,s)}(\Delta) \to \mathbb{R}^{2(r+s)+|F|}$ (see example 2.12).*

Then the forgetful map ft *of definition 2.15 gives a bijection between oriented and unoriented (r,s)-marked Welschinger curves through $\mathscr{P}$ with degree Δ and set of fixed ends F.*

Proof. As in proposition 3.3, we have to prove three statements.

(a) ft maps oriented to unoriented Welschinger curves through $\mathscr{P}$: Let $C \in M_{(r,s)}^{\text{or}}(\Delta, F)$ be an oriented Welschinger curve. The list of allowed vertex types for C implies that C satisfies condition (i) of definition 4.6. Condition (ii) follows from lemma 4.9.

(b) ft is injective on the set of curves through $\mathscr{P}$: Notice that under the end-gluing map of definition 4.1, a vertex of type (8) becomes a vertex of type (4), and type (6b) becomes (7). Thus the image C' under the end-gluing map satisfies the conditions of lemma 2.13 (b) by lemma 2.21 and remark 4.3. Lemma 2.20 implies that there is at most one possible orientation on C', and it follows immediately that there is only one possible orientation on C, since double ends have to point outwards (types (6b) and (8)).

(c) ft is surjective on the set of curves through $\mathscr{P}$: Let $C \in M_{(r,s)}(\Delta)$ be an unoriented Welschinger curve through $\mathscr{P}$ with set of fixed ends F. Let α be the combinatorial type of C and $M_{(r,s)}^{\alpha}(\Delta)$ its corresponding cell in $M_{(r,s)}(\Delta)$. Denote by s_1 the number of isolated complex markings in Γ_{even}, and by k the number of double ends. As this means by definition 4.1 and condition (i) that there are at least $s_1 + k$ vertices of valence 4 it follows from [GM08] proposition 2.11 that the dimension of $M_{(r,s)}^{\alpha}(\Delta)$ is at most $|\Delta| + r + s - 1 - s_1 - k = 2r + 3s + |F| - s_1 - k$. On the other hand, C passes through a collection of conditions in general position, so $\dim(M_{(r,s)}^{\alpha}(\Delta)) \geq 2r + 2s + |F|$. It follows that

$$s - s_1 - k \geq 0. \tag{$*$}$$

In fact, we want to show that we always have equality here. For this let $\tilde{\Gamma}$ be a connected component of $\Gamma_{\text{even}} \backslash (\overline{\Gamma \backslash \Gamma_{\text{even}}})$ — i.e. we remove from Γ_{even} all attachment vertices to its complement — which is not an isolated marked end. Denote by $\tilde{\Gamma}'$ its image under the end-gluing map. Let $\tilde{s}$ be the number of complex markings belonging to $\tilde{\Gamma}$, and let $\tilde{k}$ be the number of its double ends. Then $\tilde{\Gamma}'$ contains possibly fixed even ends, the $\tilde{k}$ ends coming from the double ends, and one extra end (which is either the root itself or the edge with which it is adjacent to $\Gamma \backslash \Gamma_{\text{even}}$). If $\tilde{s} > \tilde{k}$ it follows that there is a component of $\tilde{\Gamma}'$ minus the $\tilde{s}$ complex markings which does not contain a non-fixed end, which would be a contradiction to lemma 2.13 (a). Thus $\tilde{s} \leq \tilde{k}$. Summing this up over all such components $\tilde{\Gamma}$ it follows that the number $s - s_1$ of complex markings which are non-isolated in Γ_{even} satisfies $s - s_1 \leq k$. Together with $(*)$ this yields $s - s_1 = k$, as desired.

Hence equality holds in all our estimates above. There are various consequences of this: first of all, we have $\dim(M_{(r,s)}^{\alpha}(\Delta)) = 2r + 2s + |F|$, and C has exactly s vertices of valence

4, namely s_1 adjacent to complex markings which are isolated in Γ_{even}, and $s - s_1$ adjacent to double ends. All other vertices have valence 3. In particular, if the root of a connected component of Γ_{even} is not an end, it has to be at a 3-valent vertex. Also, since we have $\tilde{s} = \tilde{k}$ complex markings on the components $\tilde{\Gamma}$ above, it follows that there cannot be additional real markings on these components, since otherwise there would be a connected component of $\tilde{\Gamma}'$ without the markings again which does not contain a non-fixed end. Thus there are no real markings which are non-isolated in Γ_{even}.

The combinatorial type of the image C' of C under the end-gluing map is of dimension $\dim(M^\alpha_{(r,s)}(\Delta)) = 2r + 2s + |F|$ by remark 4.3. Since C has 4-valent vertices only at complex markings resp. double ends, it follows that C' has 4-valent vertices only at complex markings, and so we can apply lemma 2.13 to C' to see that there is an orientation on C' that points on each edge towards the unique non-fixed unmarked end in $\Gamma' \setminus (x_1 \cup \cdots \cup x_{r+s})$. We can define an orientation on C by orienting double ends just as the end they map to under the end-gluing map.

It remains to be shown that, for this orientation of C, we only have the vertex types (1) to (5), (6b), (7) or (8). As in the proof of proposition 3.3 (c), all edges adjacent to a vertex V point outwards if there is a marking at V, and exactly one points outwards otherwise. It is impossible that exactly one edge at V is odd. We have seen that V is 4-valent if it is adjacent to a double end, or to a complex marking, and 3-valent otherwise. The only vertex types compatible with all these restrictions are the types (1) to (8), and the three special ones in the picture of the proof of proposition 3.3 (c). Type (6a) cannot appear since each root has to be 3-valent by the above. The left picture in the proof of proposition 3.3 (c) is excluded since there are no non-isolated real markings in Γ_{even}. The middle picture is excluded since we have 4-valent vertices only at isolated complex markings or double ends. The right picture would be a root of a component $\tilde{\Gamma}$ as above. But because of the orientation there is no connection from this vertex via one of the odd edges to a non-fixed unmarked end without passing a marking. With $\tilde{k}$ non-fixed ends and $\tilde{k}$ complex markings in $\tilde{\Gamma}$ this would again lead to a connected component of Γ minus the markings with no non-fixed end, a contradiction to lemma 2.13 (a).

$\square$

Remark 4.11 (Unoriented Welschinger curves after end-gluing). In addition to definition 4.6 (b) we can also describe unoriented Welschinger curves after the end-gluing: fix a degree $\Delta = (v_1, \ldots, v_n)$ and $F \subset \{1, \ldots, n\}$. We then allow curves of any degree $\Delta' = ((v(y_i) : i \neq i_1, j_1, \ldots, i_k, j_k), (2 \cdot v(y_{i_1}), \ldots, 2 \cdot v(y_{i_k}))$ for some $i_1 < j_1, \ldots, i_k < j_k$ in $\{1, \ldots, n\} \setminus F$ such that the unmarked ends y_{i_l} and y_{j_l} have the same odd direction. For a curve $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_{n-k}, h) \in M_{(r,s)}(\Delta')$, we define Γ_{even} as in definition 3.1 as the subgraph of all even edges. We then require that complex markings are adjacent to 4-valent vertices, or non-isolated in Γ_{even}; and that each connected component of Γ_{even} has a unique root. An example of such an unoriented Welschinger curve after end-gluing is the curve (A) in the introduction.

Now we define enumerative numbers of Welschinger curves. As for broccoli curves, we work with oriented Welschinger curves from now on, keeping in mind that it does not matter whether we count oriented or unoriented Welschinger curves by proposition 4.10.

Notation 4.12. Let $r + 2s + |F| = |\Delta| - 1$, and denote by $M^W_{(r,s)}(\Delta, F)$ the closure of the space of all Welschinger curves in $M^{\text{or}}_{(r,s)}(\Delta, F)$. This is obviously a polyhedral subcomplex. By lemma 2.21 it is of pure dimension $2(r+s) + |F|$, and its maximal open cells correspond exactly to the Welschinger curves in $M^W_{(r,s)}(\Delta, F)$. For $F = \emptyset$ we write $M^W_{(r,s)}(\Delta, F)$ also as $M^W_{(r,s)}(\Delta)$.

Definition 4.13 (Welschinger numbers). Let $r, s \geq 0$, let $\Delta = (v_1, \ldots, v_n)$ be a collection of vectors in $\mathbb{Z}^2 \backslash \{0\}$, and let $F \subset \{1, \ldots, n\}$ such that $r + 2s + |F| = |\Delta| - 1$. Moreover, let $\mathscr{P} \in \mathbb{R}^{2(r+s)+|F|}$ be a collection of conditions in general position for Welschinger curves, i.e. for the evaluation map $\mathrm{ev}_F : M^W_{(r,s)}(\Delta, F) \rightarrow \mathbb{R}^{2(r+s)+|F|}$. Then we define the *Welschinger number*

$$N^W_{(r,s)}(\Delta, F, \mathscr{P}) := \frac{1}{|G(\Delta, F)|} \cdot \sum_C m_C,$$

where the sum is taken over all Welschinger curves C in with degree Δ, set of fixed ends F, and $\mathrm{ev}(C) = \mathscr{P}$. As in the case of broccoli invariants, the group $G(\Delta, F)$ compensates for the overcounting of curves due to relabeling the non-fixed unmarked ends (see remark 4.18). The sum is finite by the dimension statement of notation 4.12, and the multiplicity m_C is as in definition 2.17. For $F = \emptyset$ we abbreviate the numbers as $N^W_{(r,s)}(\Delta, \mathscr{P})$.

In contrast to the broccoli invariants of definition 3.5 we will see in remark 4.24 that these Welschinger numbers will in general depend on the choice of conditions $\mathscr{P}$.

Example 4.14 (Welschinger numbers for degrees with non-fixed even ends). In two special cases when the degree $\Delta = (v_1, \ldots, v_n)$ contains one or several non-fixed even ends we can actually compute the Welschinger numbers directly:

(a) Assume that Δ contains more than one non-fixed even end.

Consider a Welschinger curve $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h)$ contributing to the number $N^W_{(r,s)}(\Delta, F, \mathscr{P})$. Every even non-fixed end belongs to a connected component of Γ_{even} and is a root. Since every connected component has a unique root by definition 4.6 (b) (ii) it follows that such a component cannot meet the remaining part $\overline{\Gamma \backslash \Gamma_{\text{even}}}$. But as the curve is connected this means that Γ_{even} can have only one connected component and thus only one root. This is a contradiction, showing that there is no Welschinger curve with more than one non-fixed even end, and thus that in this case

$$N^W_{(r,s)}(\Delta, F, \mathscr{P}) = 0.$$

(b) Assume now that Δ contains exactly one non-fixed end of weight 2, of direction v_1, and only non-fixed edges of weight 1 otherwise.

Assume that $N^W_{(r,s)}(\Delta, \mathscr{P}) \neq 0$. By the same argument as in (a) each curve contributing to $N^W_{(r,s)}(\Delta, \mathscr{P})$ is totally even (containing one even and $\frac{|\Delta|-1}{2}$ double ends). Hence $|\Delta|$ must be odd and must contain each vector v_i $(i \neq 1)$ twice. Without restriction we can assume that $v_i = v_{i + \frac{|\Delta|-1}{2}}$ for $1 < i \leq \frac{|\Delta|-1}{2} + 1$. Furthermore, it then follows that $r = 0$ and $s = \frac{|\Delta|-1}{2}$.

In other words, each curve contributing to $N^W_{(0,s)}(\Delta, \mathscr{P})$ contains only vertices of type (4), (6b), (7), and (8). We can thus interpret the number $N^W_{(0,s)}(\Delta, \mathscr{P})$ as a "double complex enumerative number" in the following sense: let $\Delta' = (\frac{1}{2}v_1, v_2, \ldots, v_{\frac{|\Delta|-1}{2}+1})$ and denote by $N^C_s(\Delta', \mathscr{P})$ the number of (3-valent) tropical curves (without labeled ends) passing through $\mathscr{P}$ as e.g. in [GM07b], i.e. each curve is counted with its usual complex multiplicity as in [Mik05]. If we forget the labels of the non-marked ends, the set of curves contributing to $N^W_{(0,s)}(\Delta, \mathscr{P})$ is then obviously in bijection to the set of curves contributing to $N^C_s(\Delta', \mathscr{P})$ by multiplying each direction vector (after end-gluing) with $\frac{1}{2}$. However, $N^W_{(0,s)}(\Delta, \mathscr{P})$ is not quite equal to $N^C_s(\Delta', \mathscr{P})$ since the multiplicities of the curves are slightly different:

- If the vector $\frac{1}{2}v_1$ occurs d times in Δ' then there are d choices in the count of $N^W_{(0,s)}(\Delta, \mathscr{P})$ which of the ends of the "double complex curve" is the weight-2 end of the Welschinger curve.

- As we count Welschinger curves with labeled ends to get the number $N^W_{(0,s)}(\Delta, \mathscr{P})$, we overcount each curve without labeled ends by a factor of $|G(\Delta)| \cdot 2^{-\frac{|\Delta|-1}{2}}$ (see remark 4.18), since $\frac{|\Delta|-1}{2}$ is the number of double ends.

- Under the bijection, each vertex of type of type (4) and (8) maps to a vertex of complex multiplicity $\frac{a}{4}$. Denote by Γ' the graph after end-gluing and forgetting the marked points. This graph has $\frac{|\Delta|-1}{2} + 1$ ends and is 3-valent, thus we have $n_{(4)} + n_{(8)} = \frac{|\Delta|-1}{2} - 1$. Therefore we overcount each Welschinger curve by an additional factor of $4^{\frac{|\Delta|-1}{2}-1}$.

- In addition, we count each Welschinger curve with a sign, namely $i \cdot (-1)^{n_{(8)}} \cdot i^{-n_{(4)}-n_{(6b)}}$, where the factor of i arises because of the end of weight 2 and the other factors arise because of the vertex multiplicities. The number of ends of the graph Γ' equals $n_{(6b)} + n_{(7)} + 1 = \frac{|\Delta|-1}{2} + 1$, thus we have $n_{(4)} + n_{(8)} + 1 = n_{(6b)} + n_{(7)}$. Since $n_{(7)} = n_{(8)}$ by 4.8, we can conclude $n_{(4)} + 1 = n_{(6b)}$, thus the sign above equals $(-1)^{n_{(8)}} \cdot i^{-2n_{(4)}} = (-1)^{n_{(4)}+n_{(8)}} = (-1)^{\frac{|\Delta|-1}{2}-1}$.

Taking all these factors together, it follows that

$$N^W_{(0,s)}(\Delta, \mathscr{P}) = d \cdot (-1)^{\frac{|\Delta|-1}{2}-1} \cdot 2^{-\frac{|\Delta|-1}{2}} \cdot 4^{\frac{|\Delta|-1}{2}-1} \cdot N^C_s(\Delta', \mathscr{P})$$

$$= d \cdot (-1)^{\frac{|\Delta|-1}{2}-1} \cdot 2^{\frac{|\Delta|-1}{2}-2} \cdot N^C_s(\Delta', \mathscr{P}).$$

In particular, in this case $N^W_{(0,s)}(\Delta, \mathscr{P})$ does not depend on the exact position of the points $\mathscr{P}$.

We will see in example 5.21 that in some cases these results hold for broccoli invariants as well.

For $F = \emptyset$ and certain choices of the degree Δ, there exist well-known Welschinger invariants in the literature that count real rational algebraic curves through given points in the plane, and that do not depend on the choice of point conditions. We want to show now that they agree with our Welschinger numbers in these cases.

Remark 4.15 (Welschinger curves compared to [Shu06]). Notice that (unoriented) Welschinger curves where all unmarked ends are non-fixed and odd correspond precisely to the curves considered by Shustin in [Shu06] (in the way described in remark 4.11). There, unparametrized tropical curves are considered, i.e. the images $h(\Gamma)$ without the parametrizing graph Γ, and it is required that the point conditions are general enough so that the Newton subdivision dual to $h(\Gamma)$ (see [Mik05] proposition 3.11) consists only of triangles and parallelograms. In this case each such unparametrized curve can uniquely (up to the labeling of the unmarked ends) be parametrized by a graph Γ' such that the map to $\mathbb{R}^2$ identifies only finitely many points. Adding an end for each marking and reversing the end-gluing by splitting each even unmarked end into a double end then gives a graph Γ together with a map $h : \Gamma \to \mathbb{R}^2$ satisfying the conditions of definition 4.6 (b). The part $h(\Gamma_{\text{even}})$ coincides with the subgraph G in [Shu06] consisting of all the even edges; their components are connected to odd edges at exactly one vertex, the root. (Other authors consider parametrized curves and the even part G as the non-fixed locus of a certain involution on the tropical curve, from which it also follows that each connected component has one root [BM08].)

The definition of the multiplicities of these curves in [Shu06] looks at first a little different compared to our definition 2.17. We will recall it here and then show that it in fact coincides with ours.

Definition 4.16 (W-multiplicity, see [Shu06] section 2.5). Let $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h)$ be a Welschinger curve, and assume that the Newton subdivision dual to $h(\Gamma)$ (see [Mik05] proposition 3.11) consists of only triangles and parallelograms. Denote by a the number of lattice points inside triangles of this subdivision, by b the number of triangles such that all sides have even lattice length,
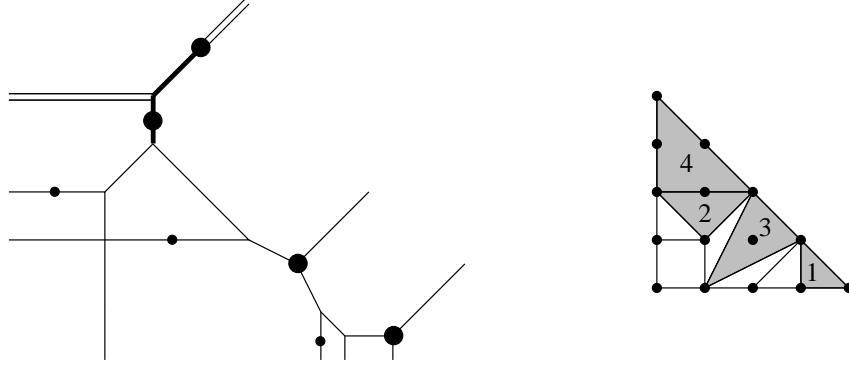
and by c the number of triangles whose lattice area is even. Then we define the *W-multiplicity* of C to be

$$\tilde{m}_C := (-1)^{a+b} \cdot 2^{-c} \cdot \prod_V \text{mult}(V),$$

where the product goes over all triangles with even lattice area or dual to vertices with a complex marking, and where $\text{mult}(V)$ denotes the integer area of this triangle, i.e. the complex vertex multiplicity as in definition 2.17.

For an unparametrized curve $h(\Gamma)$, this coincides with the definition of multiplicity in [Shu06] section 2.5.

Example 4.17. The following picture shows a Welschinger curve (without orientation) and its dual Newton subdivision. The triangles V contributing to $\tilde{m}_C$ are shaded and labeled with their integer area; we have $\tilde{m}_C = (-1)^{1+1} \cdot 2^{-2} \cdot 4 \cdot 2 \cdot 3 \cdot 1 = 6$.

Remark 4.18 (Labeled and unlabeled curves). Note that we consider curves with labeled unmarked ends, whereas the unparametrized curves in [Shu06] come without this data. Thus we overcount each unparametrized curve by a factor that records the different ways to label the (non-fixed) unmarked ends so that we get different parametrized curves. If k denotes the number of double ends then this overcounting factor is $|G(\Delta)| \cdot 2^{-k}$, where the 2^{-k} term arises because exchanging the two labels of a double end does not change the parametrized curve.

Lemma 4.19 (Multiplicity and W-multiplicity). *Let $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h)$ be a Welschinger curve of degree Δ with no fixed ends, satisfying $\omega(y_i) = 1$ for all $i = 1, \ldots, n$, and passing through points in general position as in example 2.12. Then the multiplicity m_C and the W-multiplicity $\tilde{m}_C$ of C are related by $m_C = 2^k \cdot \tilde{m}_C$, where k is the number of double ends of C.*

Proof. It follows from the list of allowed vertex types and their multiplicities that a vertex V contributes a factor of $\text{mult}(V)$ to m_C if and only if V is adjacent to a complex marking or dual to a triangle with even lattice area.

The number c of triangles with even lattice area equals $n_{(3)} + n_{(4)} + n_{(8)}$. Let $\tilde{\Gamma}$ be a connected component of Γ_{even}. We know that $\tilde{\Gamma}$ has a unique root. Since $\omega(y_i) = 1$ for all $i = 1, \ldots, n$, this root cannot be an end of Γ, so it has to be a vertex of type (3) in Γ, i.e. a 1-valent vertex in Γ_{even}. Remove the 1-valent vertex from $\tilde{\Gamma}$, thus producing an end, apply the end-gluing map of definition 4.1, and forget all markings (straightening the 2-valent vertices). Call the resulting graph $\tilde{\Gamma}^\circ$. This graph is 3-valent and has $1 + n_{(6b)}^{\tilde{\Gamma}} + n_{(8)}^{\tilde{\Gamma}}$ ends, and thus it has $n_{(6b)}^{\tilde{\Gamma}} + n_{(8)}^{\tilde{\Gamma}} - 1$ vertices. But this number of 3-valent vertices also equals $n_{(4)}^{\tilde{\Gamma}} + n_{(8)}^{\tilde{\Gamma}}$, and so $n_{(6b)}^{\tilde{\Gamma}} + n_{(8)}^{\tilde{\Gamma}} = n_{(4)}^{\tilde{\Gamma}} + n_{(8)}^{\tilde{\Gamma}} + 1 = n_{(4)}^{\tilde{\Gamma}} + n_{(8)}^{\tilde{\Gamma}} + n_{(3)}^{\tilde{\Gamma}}$. Since this holds for any $\tilde{\Gamma}$, it follows that $n_{(6b)} + n_{(8)} = n_{(3)} + n_{(4)} + n_{(8)}$. Thus $k = c$, where k denotes the number of double ends. The factor 2^k in the lemma thus corresponds exactly to the factor 2^{-c} in the definition 4.16 of $\tilde{m}_C$.

Hence it only remains to show that $(-1)^{a+b}$ equals the sign contribution coming from factors of i in the definition 2.17 of m_C, where a denotes the number of lattice points in the interior of triangles and b denotes the number of triangles such that all sides have even lattice length. We refer to the power of i in the vertex multiplicity m_V of definition 2.17 as the sign.

Consider a vertex V and let $A = \text{mult}(V)$. If V is of type (2) to (5), assume the three adjacent (non-marked) edges have weights ω_1, ω_2 and ω_3. By Pick's formula, $A = 2I + B - 2$, where I denotes the number of lattice points in the interior of the triangle dual to V and B denotes the number of lattice points on the boundary. By our assumptions, $B = \omega_1 + \omega_2 + \omega_3$. If V is of type (2) or (5), then its sign is

$$i^{A-1} = (-1)^{\frac{A-1}{2}} = (-1)^{\frac{2I+\omega_1+\omega_2+\omega_3-2-1}{2}} = (-1)^I \cdot (-1)^{\frac{\omega_1-1}{2}} \cdot (-1)^{\frac{\omega_2-1}{2}} \cdot (-1)^{\frac{\omega_3-1}{2}}.$$

If V is of type (3), its sign is

$$i^{A-1} = i^{-1} \cdot i^A = i^{-1} \cdot (-1)^{\frac{A}{2}} = i^{-1} \cdot (-1)^{\frac{2I+\omega_1+\omega_2+\omega_3-2}{2}}$$
$$= i^{-1} \cdot (-1)^I \cdot (-1)^{\frac{\omega_1-1}{2}} \cdot (-1)^{\frac{\omega_2-1}{2}} \cdot (-1)^{\frac{\omega_3}{2}},$$

where we assume that ω_3 is the even weight. For type (4), we get

$$i^{A-1} = i^{-1} \cdot i^A = i^{-1} \cdot (-1)^{\frac{A}{2}} = i^{-1} \cdot (-1) \cdot (-1)^{\frac{2I+\omega_1+\omega_2+\omega_3}{2}}$$
$$= i^{-1} \cdot (-1) \cdot (-1)^I \cdot (-1)^{\frac{\omega_1}{2}} \cdot (-1)^{\frac{\omega_2}{2}} \cdot (-1)^{\frac{\omega_3}{2}}.$$

We write the sign of type (6b) as $i^{-1} = i \cdot (-1) = i \cdot (-1)^{\frac{2}{2}}$, and 2 is the weight of the even adjacent edge (since the double ends are of weight 1 by assumption). The sign of (8) is

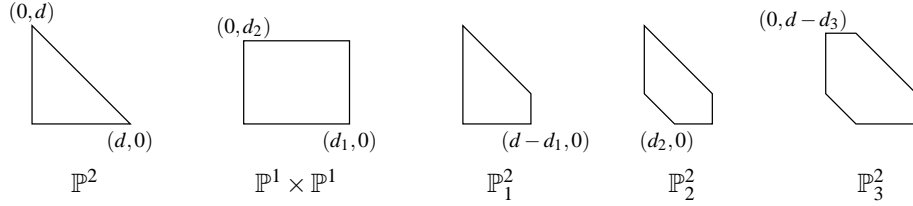$$-1 = (-1) \cdot i^A = (-1) \cdot (-1)^I \cdot (-1)^{\frac{\omega_1}{2}} \cdot (-1)^{\frac{\omega_2}{2}},$$

where ω_1 and ω_2 are the weights of the two adjacent even edges. This is true since the two edges of the same direction which are adjacent to (8) are ends and thus their weight is 1 by assumption. The sign of (1) can be written as $1 = (-1)^{\frac{\omega_1-1}{2}} \cdot (-1)^{\frac{\omega_2-1}{2}}$, where $\omega_1 = \omega_2$ is the odd weight of the adjacent edges. Analogously, we can write the sign of (7) as $1 = (-1)^{\frac{\omega_1}{2}} \cdot (-1)^{\frac{\omega_2}{2}}$, where now $\omega_1 = \omega_2$ is the even weight of the adjacent edges.

Notice that the product of the factors $(-1)^I$ which appear for each vertex dual to a triangle is $(-1)^a$. Also, for each vertex of type (4) and (8) — which are the vertices dual to triangles such that all sides have even lattice length — we have a factor of (-1) which yields $(-1)^b$ as product. In addition, we have extra factors of i^{-1} for each vertex of type (3) and (4), and i for each vertex of type (6b). But since $n_{(4)} + n_{(3)} = n_{(6b)}$ as we have seen above, these extra factors cancel. Furthermore, we have factors of $(-1)^{\frac{\omega-1}{2}}$ for each edge of odd weight ending at a vertex, and $(-1)^{\frac{\omega}{2}}$ for each even edge. Every bounded edge ends at two vertices, so these contributions cancel. Since we require that the weights of all ends are 1, the corresponding factors for the ends are just 1. Thus all the factors $(-1)^{\frac{\omega-1}{2}}$ resp. $(-1)^{\frac{\omega}{2}}$ cancel, and it follows that the sign equals $(-1)^{a+b}$, as required. $\qquad\square$

Remark 4.20 (Welschinger numbers compared to [Shu06]). It follows from remark 4.15, remark 4.18, and lemma 4.19 that for $F = \emptyset$ and Δ consisting of primitive vectors (i.e. of directions of weight one) our Welschinger number $N^W_{(r,s)}(\Delta, \mathscr{P})$ of definition 4.13 equals the number of unparametrized curves as in [Shu06], counted with their W-multiplicities as in definition 4.16.

Remark 4.21 (Algebraic Welschinger invariants). To see the enumerative meaning of the Welschinger numbers let us now discuss a Correspondence Theorem stating that our tropical count determines the algebraic Welschinger invariants, i.e. numbers of real rational curves passing through a set of conjugation invariant points, counted with weight ± 1 according to the nodes. More precisely, let Σ be a real toric unnodal Del Pezzo surface and D a real ample linear system on Σ. There are five such surfaces, namely $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ blown up at $k \leq 3$ generic real points (denoted by $\mathbb{P}^2_k$), equipped

with the standard real structure. The linear system D is in suitable toric coordinates generated by monomials $x^i y^j$, where (i,j) ranges over all lattice points of a polygon $Q(D)$ of the following form. If $\Sigma = \mathbb{P}^2$ and D is the class of d times a line, then $Q(D)$ is the triangle with vertices $(0,0)$, $(d,0)$, and $(0,d)$. If $\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$ and D is of bidegree (d_1,d_2) then $Q(D)$ is the rectangle with vertices $(0,0)$, $(d_1,0)$, (d_1,d_2), and $(0,d_2)$. If $\Sigma = \mathbb{P}^2_k$ and $D = d \cdot L - \sum_{i=1}^k d_i E_i$, (where L denotes the class of the pull-back of a line, and E_i denote the exceptional divisors of $\mathbb{P}^2_k \to \mathbb{P}^2$), then $Q(D)$ is the trapezoid with vertices $(0,0)$, $(d-d_1,0)$, $(d-d_1,d_1)$, $(0,d)$ if $k=1$, the pentagon with vertices $(d_2,0)$, $(d-d_1,0)$, $(d-d_1,d_1)$, $(0,d)$, $(0,d_2)$ if $k=2$, and the hexagon with vertices $(d_2,0)$, $(d-d_1,0)$, $(d-d_1,d_1)$, $(d_3,d-d_3)$, $(0,d-d_3)$, $(0,d_2)$ if $k=3$.

$$Q(D): \qquad \mathbb{P}^2 \qquad \mathbb{P}^1 \times \mathbb{P}^1 \qquad \mathbb{P}^2_1 \qquad \mathbb{P}^2_2 \qquad \mathbb{P}^2_3$$

Let r and s be non-negative integers satisfying $\#(\partial Q(D) \cap \mathbb{Z}^2) - 1 = r + 2s$, and let $\mathscr{P}$ be a generic conjugation invariant set of $r + 2s$ points of which exactly r points are conjugation invariant themselves. By the Welschinger theorem ([Wel03], [Wel05]), the set $\mathscr{R}(\Sigma, D, \mathscr{P})$ of algebraic real rational curves $C \in D$ passing through $\mathscr{P}$ is finite, consists only of nodal and irreducible curves, and the number

$$W_\Sigma(D,r,s) := \sum_{C \in \mathscr{R}(\Sigma,D,\mathscr{P})} (-1)^{s(C)}$$

called *Welschinger invariant* does not depend on the special choice of $\mathscr{P}$, where $s(C)$ denotes the number of solitary nodes of C, i.e. real points where the curve is locally given by the equation $x^2 + y^2 = 0$.

Definition 4.22 (Toric Del Pezzo degrees). We say that a degree Δ is *toric Del Pezzo* if it consists of the primitive normal directions of facets of one of the polytopes $Q(D)$ of remark 4.21, where each direction appears l times if l is the lattice length of the corresponding facet. If $Q(D)$ is the triangle with endpoints $(0,0)$, $(d,0)$ and $(0,d)$ (corresponding to the class of d times a line in $\mathbb{P}^2$), then we call curves of degree Δ consisting of the normal directions $(-1,0)$, $(0,-1)$ and $(1,1)$ each d times *curves of degree d*.
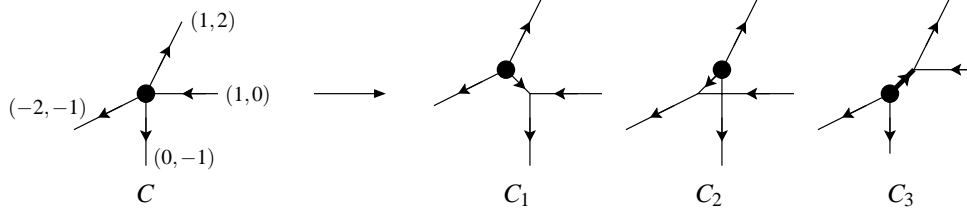
Notice that a toric Del Pezzo degree consists of directions of weight one, so the requirements of lemma 4.19 are satisfied.

Theorem 4.23 (Correspondence Theorem). *Let Σ be a toric Del Pezzo surface, D a real ample linear system, $Q(D)$ the corresponding polytope as in remark 4.21, and Δ the corresponding degree. Let r and s satisfy $|\Delta| - 1 = \#(\partial Q(D) \cap \mathbb{Z}^2) - 1 = r + 2s$. Then $N^W_{(r,s)}(\Delta, \mathscr{P}) = W_\Sigma(D,r,s)$ for any choice of points $\mathscr{P}$ in general position. In particular, the Welschinger numbers $N^W_{(r,s)}(\Delta, \mathscr{P})$ are independent of $\mathscr{P}$ in this case.*

Proof. Using remark 4.20, this is theorem 3.1 of [Shu06]. □

Remark 4.24 (Welschinger numbers are not locally invariant in the moduli space). It is a striking feature of the Welschinger numbers $N^W_{(r,s)}(\Delta, \mathscr{P})$ that, although they are invariant under $\mathscr{P}$ in the cases mentioned in theorem 4.23, one cannot show this by a local study of the moduli space as in the proof of theorem 3.6. In short, the reason for this is that the absence of the vertex type (6a) breaks the local invariance argument in the codimension-1 case (C1$_1$) (see the proof of theorem 3.6, in particular the table of codimension-1 cases and their resolutions).

For example, consider a combinatorial type corresponding to a cell of $M^W_{(r,s)}(\Delta)$ of codimension one which locally contains the left picture C below:
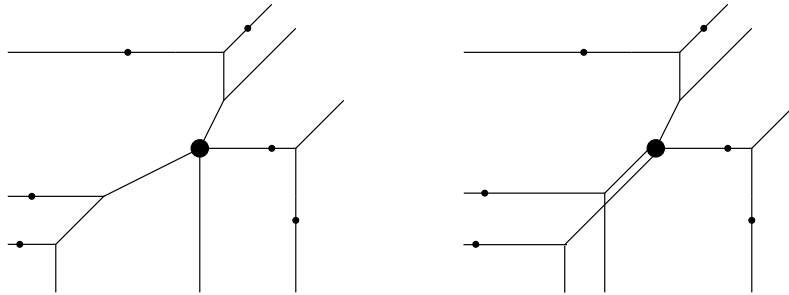

Curves of this type pass through conditions which are not in general position, since the horizontal edge is fixed and the complex point is exactly on this horizontal line. There are two Welschinger curves C_1 and C_2 as in the picture above such that this type appears in their boundary. Their multiplicities are $m_{C_1} = i^0 \cdot 3i^2 = -3$ and $m_{C_2} = i^0 \cdot 1 \cdot i^0 = 1$. We can see that they both satisfy the conditions when we move the complex point above the horizontal line. In contrast, no Welschinger curve satisfies the conditions if we move the point below the line: the third resolution C_3 would require a vertex of type (6a), which is not allowed for Welschinger curves. Thus locally around this codimension-1 cone, the number of Welschinger curves is not invariant.
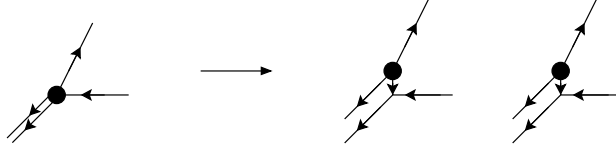
Of course, this leads to choices of Δ for which the Welschinger numbers are not invariant. For example, we can pick $\Delta = ((1,0),(0,-1),(-2,-1),(1,2))$ such that the picture above is actually a global picture. Then this example shows that $N^W_{(r,s)}(\Delta, \mathscr{P}) = -2$ if we pick $\mathscr{P}$ with the complex point above the horizontal line, and $N^W_{(r,s)}(\Delta, \mathscr{P}) = 0$ if we pick $\mathscr{P}$ with the complex point below the line. Thus the numbers depend on the choice of $\mathscr{P}$ and are not invariant.

However, if Δ is a toric Del Pezzo degree as in definition 4.22, then it follows from the Correspondence Theorem 4.23 (and the Welschinger theorem) that the numbers $N^W_{(r,s)}(\Delta, \mathscr{P})$ are invariant.

Since this is true in spite of the missing local invariance around codimension-1 cones we can observe the following interesting fact about the moduli space $M^W_{(r,s)}(\Delta)$ and the map ev: given a collection of points $\mathscr{P}$ not in general position such that a curve of a codimension-1 type is in the preimage $\mathrm{ev}^{-1}(\mathscr{P})$ for which we do not have local invariance (as for the example above), there must be another curve in $\mathrm{ev}^{-1}(\mathscr{P})$ which is also of a codimension-1 type not satisfying local invariance, such that the differences to the invariance cancel exactly. For example, if we consider the above example as a local picture of the curve of degree 3 below, then there is a second curve of codimension 1 such that the two differences cancel. The following picture shows these two codimension-1 curves passing through $\mathscr{P}$ not in general position:

We have seen already that the left picture produces a local difference of -2: locally, the difference between the numbers of curves passing through the configuration where we move the complex point up and down is -2. The right picture now produces a local difference of $+2$:
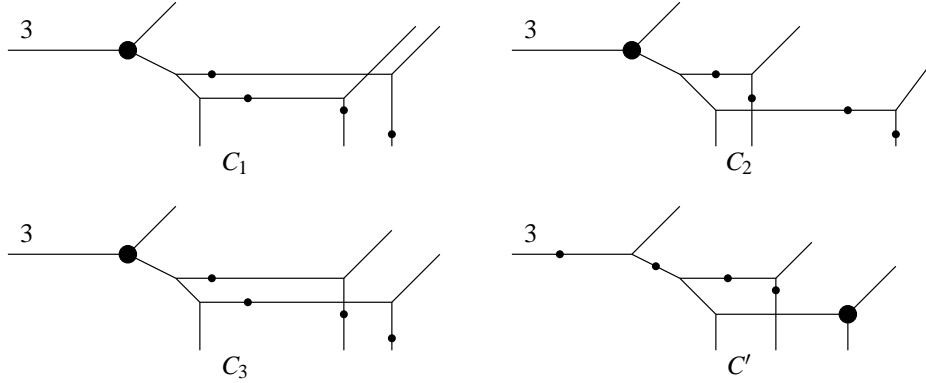
There are again two Welschinger curves which have this codimension one curve in their boundary (notice that the two edges pointing to the bottom-left are distinguishable in the big picture). They both satisfy the conditions when the complex point is moved up. No Welschinger curve satisfies the conditions if the complex point is moved down. Their multiplicity is $i^0 \cdot 1 \cdot i^0 = 1$ each.

If the degree Δ is not a toric Del Pezzo degree, in particular if Δ contains non-primitive vectors (i.e. we consider *relative* Welschinger numbers), it may happen that these numbers are not even globally invariant. This has already been observed in [ABLdM10] with the following example.

Example 4.25 (Welschinger numbers are in general not invariant, see [ABLdM10] section 7.2). The following picture shows the three Welschinger curves C_1, C_2, C_3 (up to relabeling of the unmarked ends) of degree

$$((-3,0),(0,-1),(0,-1),(0,-1),(1,1),(1,1),(1,1))$$

passing through some given configuration $\mathscr{P}$ of points. Each counts with multiplicity 3, so for this configuration we have $N_{(r,s)}^W(\Delta, \mathscr{P}) = 9$. For the configuration on the bottom right however, there is only one Welschinger curve C' passing through it, and it is of multiplicity one. So in this case $N_{(r,s)}^W(\Delta, \mathscr{P}') = 1$, i.e. the number depends on the choice of $\mathscr{P}$.

5. BRIDGE CURVES

The aim of the following section is to prove that for toric Del Pezzo degrees Δ (see definition 4.22) the Welschinger numbers $N_{(r,s)}^W(\Delta, \mathscr{P})$ coincide with the broccoli invariants $N_{(r,s)}^B(\Delta, \mathscr{P})$ (see corollary 5.16). Since broccoli invariants are independent of the chosen conditions, this result provides a tropical proof of the invariance of Welschinger numbers, without having to use the detour via the Correspondence and the Welschinger theorem. When considering degrees Δ that are not toric Del Pezzo, the equivalence of Welschinger numbers and broccoli invariants no longer holds, and consequently the Welschinger numbers may actually not be invariant.

We start with the definition of the class of bridge curves. It is a special case of the class of oriented marked curves and includes oriented broccoli and Welschinger curves. When a bridge curve is a broccoli curve having vertices of type (6a) or a Welschinger curve having vertices of type (8), this curve allows to start a so called *bridge*, that is, a 1-dimensional family of bridge curves connecting broccoli and Welschinger curves. We show the invariance of the curve multiplicities m_C along these bridges, which then leads to the equality of broccoli and Welschinger numbers mentioned above.

Throughout this section let $r, s \geq 0$, let $\Delta = (v_1, \ldots, v_n)$ be a collection of vectors in $\mathbb{Z}^2 \setminus \{0\}$, and let $F \subset \{1, \ldots, n\}$ such that $|\Delta| - 1 = r + 2s + |F|$. Moreover, fix conditions $\mathscr{P} \in \mathbb{R}^{2(r+s)+|F|}$ in general position for $\mathrm{ev}_F : M^{\mathrm{or}}_{(r,s)}(\Delta, F) \to \mathbb{R}^{2(r+s)+|F|}$ as in definition 2.10 and example 2.12, and consider only curves satisfying these conditions.

Remark 5.1. Note that by lemma 2.21 an oriented curve $C \in M^{\mathrm{or}}_{(r,s)}(\Delta, F)$ all of whose vertices are of the types (1) to (9) of definition 2.17 satisfies $n_{(7)} = n_{(8)} + n_{(9)}$ (similarly to remark 4.8 (a) for Welschinger curves).

Definition 5.2 (Bridge curves). Let r, s, Δ, and F be as in remark 5.1. A *bridge curve* consists of the data of:

- an oriented curve $C \in M^{\mathrm{or}}_{(r,s)}(\Delta, F)$ all of whose vertices are of the types (1) to (9) of definition 2.17, and

- a bijection between its vertices of type (7) and those of types (8) or (9) (see remark 5.1),
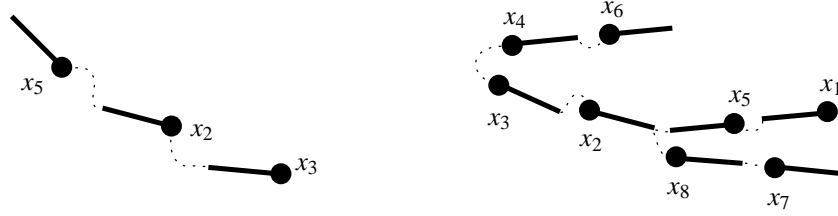
such that the following conditions hold:

(a) There is at most one vertex of type (9).

(b) Each vertex of type (8) or (9) is connected to its corresponding vertex of type (7) (under the given bijection) starting with one of its even edges by a sequence of edges with no markings on them.

(c) Consider the set M of vertices of type (6a) and (7); by abuse of notation we will sometimes also think of it as the set of all complex markings at these vertices. We split this set as $M = M_{(8)} \dot\cup M_{(9)} \dot\cup M_{(6a)}$, where

- $M_{(8)}$ contains the vertices of type (7) corresponding to vertices of type (8) under the given bijection,

- $M_{(9)}$ contains the vertices of type (7) corresponding to vertices of type (9) under the given bijection,

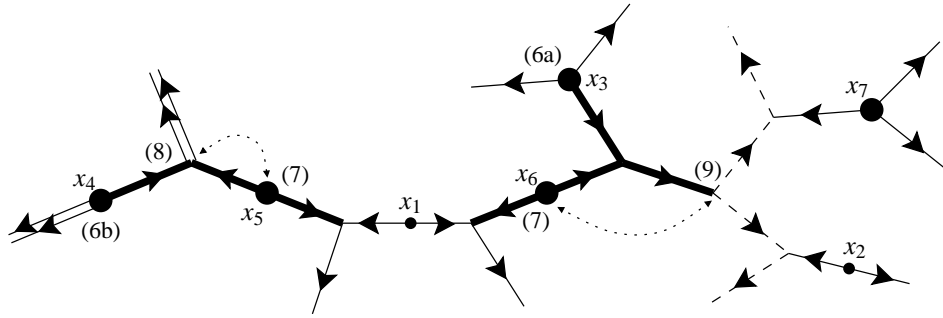- $M_{(6a)}$ contains the vertices of type (6a).

We define a partial order on M by considering each vertex in M with one even adjacent edge — in the case of a vertex of type (7) we take the edge that does not connect this vertex to its corresponding vertex of type (8) or (9). For complex markings $x_i \neq x_j$ in M we say $x_i < x_j$ if the unique path connecting x_i and x_j does not pass through the even edge of x_i, but does pass through the even edge of x_j. Refine this partial order to a total order by considering vertices which are minimal under the partial order and comparing the (numerical) value of their markings. Choose the numerically minimal one and repeat the procedure without the chosen vertex until all vertices are ordered. We require now that the labeling of the complex markings is chosen such that vertices in $M_{(8)}$ are smaller than vertices in $M_{(9)}$, and vertices in $M_{(9)}$ are smaller than vertices in $M_{(6a)}$.

The multiplicity m_C of a bridge curve C is given as usual by definition 2.17.

Example 5.3. For an example of the partial order in definition 5.2 (c) consider the picture below on the left, in which x_2, x_3, and x_5 are the complex markings of type (6a) or (7). We have $x_5 < x_2 < x_3$, where dotted lines stand for parts of the graph between the distinguished edges and vertices. In this case, the total order on M of definition 5.2 (c) agrees with this partial order. In the picture on the right however we get the partially ordered sets $x_7 < x_8 < x_5 < x_1$, $x_7 < x_8 < x_2 < x_3$, $x_6 < x_4$, and the total order $x_6 < x_4 < x_7 < x_8 < x_2 < x_3 < x_5 < x_1$.

Example 5.4. An example of a bridge curve (containing a vertex of type (9)) is given in the following picture; the bijection between the vertices of type (7) and those of types (8) and (9) is indicated by the dotted arrows. We have labeled the vertices by their types only in the cases (6), (7), (8), and (9) since these are the most relevant ones for our study of bridge curves. In this example we have $M = \{x_3, x_5, x_6\}$ and $M_{(8)} = \{x_5\}$, $M_{(9)} = \{x_6\}$, $M_{(6a)} = \{x_3\}$. The partial order on M is given by $x_6 < x_3$ and the total order by $x_5 < x_6 < x_3$. The dashed edges are ordinary odd edges; they form a string as explained in definition 5.9 and remark 5.10.

Remark 5.5. From the allowed vertex types of definition 2.17 it follows that the sequence of edges of definition 5.2 (b) connecting each vertex of type (7) to its corresponding vertex of type (8) or (9) just contains even edges which are then adjacent to vertices of type (4).

Remark 5.6. The choice of the total order refining the partial order in definition 5.2 (c) is not important. While the definition of bridge curves depends on this choice, the result of invariance in theorem 5.14 does not.

Remark 5.7 (Dimension of the space of bridge curves). These (oriented) bridge curves can be constructed with the bridge algorithm 5.18 from oriented broccoli or Welschinger curves without changing the conditions $\mathscr{P}$. In particular, bridge curves are curves passing through conditions in general position. In fact, since the number of our conditions is $2(r + s) + |F|$ it follows from lemma 2.21 that the space of bridge curves of a given combinatorial type through $\mathscr{P}$ is 0-dimensional if there is no vertex of type (9) (i.e. if $M_{(9)} = \emptyset$), and 1-dimensional otherwise. If we even have $M_{(8)} = M_{(9)} = \emptyset$ or $M_{(9)} = M_{(6a)} = \emptyset$, the bridge curves specialize to the broccoli and Welschinger curves that we already know:

Lemma 5.8 (Broccoli and Welschinger curves as bridge curves). *For fixed r, s, Δ, F the operation of forgetting the correspondence between the vertices of type (7) and those of types (8) or (9) of definition 5.2 induces bijections between curves through $\mathscr{P}$*
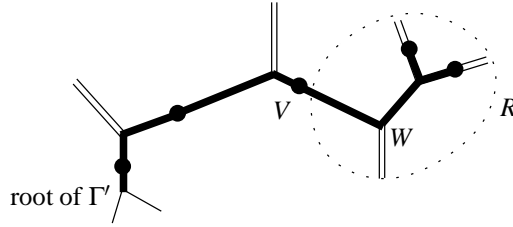
$$\{bridge\ curves\ with\ M_{(8)} = M_{(9)} = \emptyset\} \quad \overset{1:1}{\longleftrightarrow} \quad \{oriented\ broccoli\ curves\}$$

$$and \qquad \{bridge\ curves\ with\ M_{(9)} = M_{(6a)} = \emptyset\} \quad \overset{1:1}{\longleftrightarrow} \quad \{oriented\ Welschinger\ curves\}.$$

Proof. First of all, given a bridge curve with $M_{(8)} = M_{(9)} = \emptyset$, it follows directly $n_{(7)} = n_{(8)} = n_{(9)} = 0$. Hence the curve consists only of vertices of types (1) to (6) and is therefore a broccoli curve. In the same way, $M_{(9)} = M_{(6a)} = \emptyset$ for a bridge curve implies $n_{(9)} = 0$ and $n_{(6a)} = 0$ by definition 5.2 (c).

So we obtain a Welschinger curve. Hence the two maps of the lemma (from left to right) are well-defined.

Conversely, an oriented broccoli curve has only vertices of type (1) to (6). Hence $M_{(8)} = M_{(9)} = \emptyset$, and the correspondence between vertices of types (7), (8), and (9) is trivial. So the statement of the lemma about broccoli curves is obvious.

Analogously, we have $M_{(9)} = M_{(6a)} = \emptyset$ for each oriented Welschinger curve as we just allow vertices of types (1) to (5), (6b), (7), and (8). Conditions (a) and (c) of definition 5.2 are clear. So we have to prove the existence and uniqueness of a correspondence between the vertices of type (7) and (8) that satisfies (b). To do this, we perform an induction over the number $n_{(7)}$ of vertices of type (7) in the underlying graph Γ. For $n_{(7)} = 0$ there is nothing to show. Let V be such a vertex of type (7) in a connected component Γ' of Γ_{even} such that the part of $\Gamma' \setminus \{V\}$ not containing the root of Γ' (see definitions 4.4 and 4.6 (b) and the equivalence of oriented and unoriented Welschinger curves through conditions in general position in proposition 4.10) contains no other vertices of type (7). Using remark 4.8 (b) for the encircled part R in the picture below, we know that it has exactly one vertex W of type (8). Now V and W are obviously connected by a sequence of even edges as required by definition 5.2 (b), and moreover V is the only vertex of type (7) that W can be connected to without passing through other markings. Cut off R and replace V by a vertex of type (6b). Applying the induction hypothesis to the rest of Γ, we obtain the required existence and uniqueness of the bijection between the vertices of type (7) and (8).

$\square$

We will now study the 1-dimensional types of bridge curves through $\mathscr{P}$ and the boundary cases to which they can degenerate.
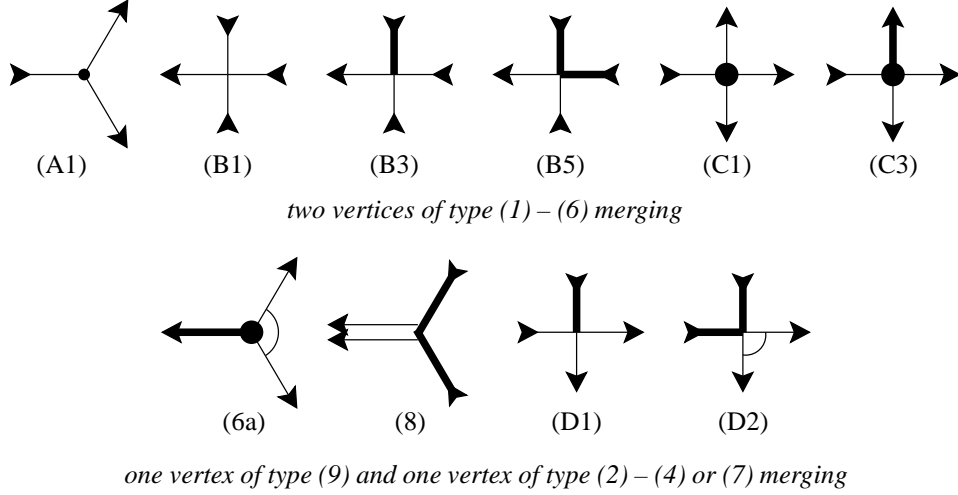
Definition 5.9 (Strings). Let $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h) \in M^{\text{or}}_{(r,s)}(\Delta, F)$ be an oriented marked curve. As in definition 3.5 (a) of [GM08], a *string* of C is a subgraph of Γ (after the end-gluing of definition 4.1) homeomorphic to $\mathbb{R}$ which does not intersect the closures $\overline{x_i}$ of the marked points and whose two ends are not fixed.

Remark 5.10. A bridge curve with a vertex of type (9) contains a unique string (containing this vertex) since the orientation of the two odd edges prescribes that they both lead in a unique way to a non-fixed unbounded end without passing through any markings (see example 5.4). Note that the allowed vertex types require that these paths to the non-fixed unbounded ends go only through vertices of types (2) and (3). In particular, the string then contains only odd edges. On the other hand, a curve without vertex of type (9) does not contain a string.

By remark 5.7, a bridge curve through conditions in general position that has a vertex of type (9) (and thus a string) moves in a 1-dimensional family — namely by moving this string, as already observed in remark 3.6 of [GM08]. Let us now figure out what boundary cases can occur at the end of such 1-dimensional families.

Lemma 5.11 (Codimension-1 cases for bridge curves). *Let C be a bridge curve through $\mathscr{P}$ with a vertex of type (9), thus having a string as in remark 5.10. This string can be moved until two vertices of C merge. The possible resulting vertices are as follows; we call them codimension-1 cases for*

bridge curves. As before, the arc in type (D2) means that the two odd edges must not be ends of the same direction.

(A1)   (B1)   (B3)   (B5)   (C1)   (C3)

two vertices of type (1) – (6) merging

(6a)   (8)   (D1)   (D2)

one vertex of type (9) and one vertex of type (2) – (4) or (7) merging

Proof. For the terminology used in the following, we refer to the proof of theorem 3.6.

Case 1: Assume the two vertices merging are of types (1) to (6). Then V is a vertex of type $(A\cdot)$, $(B\cdot)$, or $(C\cdot)$. The bridge curve we started with has already a vertex W of type (9). Hence, just resolutions that do not create a vertex of type (9) are allowed. As C originates from a bridge curve with a string, two of the edges adjacent to V are contained in the string; more precisely by remark 5.10 there must be one incoming and one outgoing odd edge. If we just consider vertices where not all resolutions have multiplicity 0, the only possible vertices which are left then are (A1), $(B1_1)$, (B3), (B5), $(C1_1)$, $(C1_3)$, and (C3).

Case 2: One vertex is of type (1) to (8) and the other one of type (7) or (8). Note that the string has to pass through one of the merging vertices in order to create the codimension-1 case. So we cannot have two vertices of type (7) and/or (8) as they do not allow the existence of the string. We thus need one vertex of type (1) to (6) which has one incoming and one outgoing odd edge, i.e. a vertex of type (3) merging with a vertex of type (7). But in this case, this vertex of type (7) (which necessarily lies in $M_{(8)}$) is bigger than the type (7) vertex in $M_{(9)}$ corresponding to the type (9) vertex at which the string starts — in contradiction to part (c) of the definition 5.2 of a bridge curve. And indeed, the vertex arising from merging type (3) with (7) has no other legal resolution, so such a case does not appear. Case 2 is thus impossible.

Case 3: One of the vertices is of type (9). Then the other vertex must be of type (2) to (4) or (7) as the other vertices of type (1), (5), (6), (8) do not fit together with the parity and the direction of the edges adjacent to the vertex of type (9).
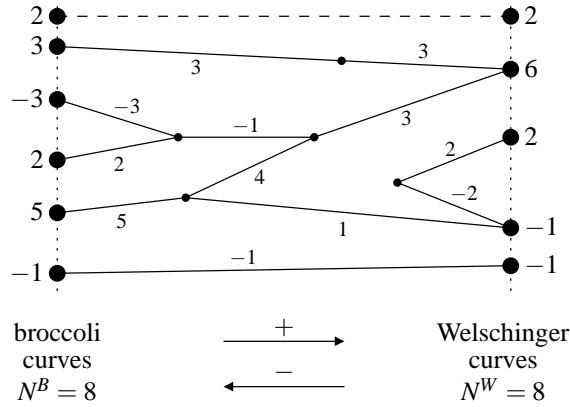
- If V arises from merging a vertex of type (9) with a vertex of type (7) we obtain a bridge curve with a vertex of type (6a), but without vertex of type (9).

- Merging a vertex of type (9) with a vertex of type (3) gives a bridge curve with a vertex of type (8) or (D2), depending on whether the resulting two odd edges are ends of the same direction or not.

- If the second vertex is of type (2) or (4), we obtain a vertex of type (D1) resp. (D2).

$\square$

Remark 5.12 (Bridge graphs and bridges). We are now able to explain the idea of bridges connecting broccoli to Welschinger curves more precisely. For this let us construct a so-called *bridge graph* as follows: the edges are the 1-dimensional types of bridge curves through $\mathscr{P}$ (i.e. those containing a vertex of type (9) and thus a string), and the vertices are their 0-dimensional boundary degenerations as described in lemma 5.11 (we will see in lemma 5.15 that in the toric Del Pezzo case the string movement actually ends at both sides and thus leads to two vertices for each edge in the bridge graph). Note that the bijection between vertices of type (7) and those of types (8) and (9) that we have for the 1-dimensional types can be extended to a map between vertices in the 0-dimensional boundary types. We identify two such 0-dimensional boundary types, i.e. represent them by the same vertex in the bridge graph, if they have the same underlying oriented curve and this map between vertices agrees, where we discard any mapping of a vertex to itself (which can occur if a type (7) vertex merges with a type (9) vertex to one of type (6a)).

Note that some vertices in the bridge graph correspond to bridge curves with no type (9) vertex, whereas others (corresponding to codimension-1 cases (A·), (B·), (C·), (D·)) are not bridge curves in the sense of our definition. Included are however (as we will see in theorem 5.14) all broccoli and Welschinger curves through $\mathscr{P}$, so that we can think of the bridge graph as connecting broccoli and Welschinger curves. We will call a connected component of the bridge graph a *bridge*.

The following picture shows a schematic example of a bridge graph. Its vertices corresponding to broccoli and Welschinger curves are drawn as big dots (on the left resp. right hand side of the diagram), the other ones as small dots. The dashed line indicates a curve which is both broccoli and Welschinger (i.e. has $M_{(8)} = M_{(9)} = M_{(6a)} = \emptyset$), so it does not correspond to an edge in the bridge graph. The broccoli and Welschinger curves, as well as the 1-dimensional types of bridge curves, are labeled with their multiplicities as in definition 2.17.

The idea to prove the equality of broccoli and Welschinger numbers is now that there is a *local balancing condition* on the bridge graph, i.e. that (as in the picture above) at each vertex the sum of the incoming equals the sum of the outgoing curve multiplicities when we move from the broccoli to the Welschinger side. To make this idea work, we first of all have to see that the edges of the bridge graph have a natural orientation so that it is well-defined which direction leads to the broccoli and which to the Welschinger side.

Definition 5.13 (Direction of string movement). For a given bridge curve C with a vertex V of type (9) consider the even edge E adjacent to V. Changing the length of E induces the movement of the string in C. Namely, making this edge longer makes the curve "more Welschinger"; we want to call this the *positive direction* (+) of the string movement. Making E shorter leads to a "more broccoli" like curve; we want to call this the *negative direction* (−) of the string movement.

Theorem 5.14 (Invariance along bridges). *Let C be an oriented curve containing a vertex V of one of the codimension-1 types (A·), (B·), (C·), (6a)/(8), or (D·) as in lemma 5.11, and only vertices of*
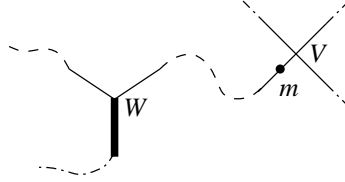
types (1) to (9) otherwise. Assume as in lemma 5.11 that C arises from moving a string in a bridge curve with a vertex of type (9). Consider all bridge curves C' that resolve C and that have matching bijections between their vertices of type (7) and those of type (8) and (9). (In the language of remark 5.12 this means that C corresponds to a vertex and C' to its adjacent edges in the bridge graph.)

The curves C' all contain a string and thus we can define $\mathrm{sign}_{C'}$ as the direction of the movement of the string away from C. Then $\sum_{C'} \mathrm{sign}_{C'} \cdot m_{C'}$ equals. . .

(a) *m_C if C is a broccoli curve (i.e. we are on the left side of the bridge graph in remark 5.12);*

(b) *$-m_C$ if C is a Welschinger curve (i.e. we are on the right side of the bridge graph);*

(c) *0 in all other cases.*

Proof. For the terminology used in the following, we refer to the proof of theorem 3.6. We consider the resolving bridge curves C' and distinguish the types of V as in lemma 5.11.

Case 1: V is a vertex of type $(A \cdot)$, $(B \cdot)$, or $(C \cdot)$ (we are then in case (c) of the theorem). We then compare the H-sign in the proof of theorem 3.6 with the direction of the string movement for C'. Imagine to put a marking m on the bounded edge adjacent to V that connects this vertex on the string to the vertex W of type (9). We know from 5.11 that V can be resolved into two vertices of types (1) to (6). As the two odd edges adjacent to W are contained in the string, the 1-dimensional movement of the marking m generated by resolving V is reflected by the 1-dimensional movement of the string and hence by varying the length of the even edge at W:

Thus the H-sign equals the sign defined by the direction of the string movement (up to the same sign for all resolutions). Since we proved $\sum_{C'} (H\text{-sign}) \cdot m_{C'} = 0$ in theorem 3.6 already, it only remains to be shown in each case that all resolving curves are actually bridge curves, i.e. satisfy the conditions (a) to (c) of definition 5.2. Condition (a) is always satisfied as we do not create a vertex of type (9).

Concerning condition (b) of the definition of a bridge curve, note that in the cases $(B \cdot)$ the connection between vertices of type (7), (8), and (9) are not modified as no vertices of type (7), (8), and (9) and no markings are involved. Hence, condition (b) is satisfied in all resolutions in this case. In the resolutions of vertices of type $(A \cdot)$ and $(C \cdot)$, no vertices of type (4) are involved, which are however necessary by remark 5.5 to connect vertices of type (7) and (8), (9). Hence, also in these cases condition (b) is satisfied in all resolutions.

Looking at condition (c) of definition 5.2, the cases $(A \cdot)$ and $(B \cdot)$ are easy to manage as no vertices of type (6a) and (7) are involved (the partition of M and the total order are not changed). For the case $(C \cdot)$ we have to go into more details.

$(C1_1)$ Resolution (I) has a supplementary vertex V of type (6). If the supplementary vertex is of type (6b), it is not contained in M and need not be considered, so let us assume that V is of type (6a). Then the set M contains one more element (lying in $M_{(6a)}$) compared to the resolutions (II) and (III). The string contains the edge v_1 and therefore, the vertex contained in $M_{(9)}$ also lies behind v_1. Hence, V is bigger than the vertex of $M_{(9)}$ under the partial order. As the total order refines the partial order condition (c) is still satisfied.

$(C1_3)$ All three resolutions contain one more vertex of type (6a) in $M_{(6a)}$ than C. But also in this case, this new vertex is bigger than the already existing vertex in $M_{(9)}$. Condition (c) is thus satisfied for all three resolutions simultaneously.

(C3) Here, there are just two resolutions with a vertex of type (6a), where each time the new bounded edge is odd. The edge v_2 is even as before, the vertex in $M_{(9)}$ lies behind v_1, so the vertex in $M_{(9)}$ and this vertex can be compared under the total order but not under the partial order. Hence, condition (c) satisfied in both cases simultaneously.

In total, we can conclude that conditions (b) and (c) are fulfilled for all resolutions (if for any).

Case 2: V is a vertex of type (6a) or (8) (note that V is a priori not unique then since C has in general several vertices of type (6a) or (8)). We want to resolve vertices in this curve such that the resolutions are bridge curves with a vertex of type (9). The other way around we can ask ourselves which vertices in a bridge curve with vertex of type (9) can be merged in order to create C. After testing all possibilities we obtain two cases:

(A) the vertex of type (9) can melt with a vertex of type (7) into a vertex of type (6a);

(B) the vertex of type (9) can melt with a vertex of type (3) into a vertex of type (8), if the odd outgoing edge of the vertex of type (3) is an end and if one of the odd outgoing edges of the vertex of type (9) is also an end of the same direction.

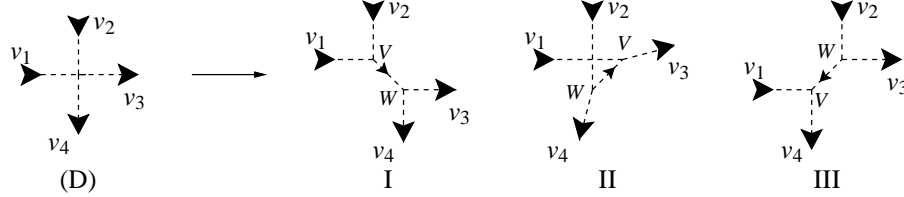Hence if we want to go the other way around, we can resolve

(A) a vertex of type (6a) into a vertex of type (7) and a vertex of type (9);

(B) a vertex of type (8) into a vertex of type (3) and a vertex of type (9). The so newly created bounded edge can have both orientations, due to the symmetric situation at the vertex of type (8). The question is just which of the vertices will become the vertex of type (3) and which one the vertex of type (9).

For these two types of resolutions we have to check if the conditions (b) and (c) of the definition 5.2 of a bridge curve are satisfied.

(A) The set M remains the same as before resolving. The connections between vertices considered in condition (b) also remain the same. Before resolving the marking is at a vertex in $M_{(6a)}$, but after resolving it becomes a vertex in $M_{(9)}$. This is just allowed if the marking was the smallest element in $M_{(6a)}$, which is the case for exactly one marking if we assume $M_{(6a)} \neq \emptyset$. Then the partial and the total order on M also remain the same and condition (c) is satisfied.

(B) The set M is conserved also in this case. Consider the marking x_i which corresponds to the vertex of type (8). In order to satisfy condition (b) of the definition we have to meet the vertex of type (9) at its even edge if we start at the marking. This means that we must choose the orientation of the inserted bounded edge such that this holds. To satisfy condition (c) the marking x_i has to be the biggest point in $M_{(8)}$ (assuming $M_{(8)} \neq \emptyset$). We need this since, after resolving the vertex, the marking lies in $M_{(9)}$ and not anymore in $M_{(8)}$. But note that we still have two resolutions as we have two possibilities to enumerate the two odd edges at the vertex of type (8) that we resolve.

Observe that both the multiplicity of the curve in (A) and the sum of the multiplicities of the two resolutions from (B) equal the multiplicity of C — due to the fact that the multiplicity of the vertex of type (8) resolved in (B) is the double of the multiplicity of the vertex of type (3) after the resolution. Thus, as the even edge E adjacent to the type (9) vertex becomes longer in (A) and shorter in the resolutions (B), the invariance holds if $M_{(8)} \neq \emptyset \neq M_{(6a)}$ so that both cases (A) and (B) exist. If $M_{(8)}$ is empty, the bridge curve we are looking at is a broccoli curve by lemma 5.8. We then resolve a vertex of type (6a) by making E longer. Hence $\text{sign}_{C'} \cdot m_{C'}$ is plus the broccoli multiplicity. In the same way, if $M_{(6a)}$ is empty, the considered bridge curve is a Welschinger curve by lemma 5.8. As we then resolve a vertex of type (8), E becomes shorter, so $\text{sign}_{C'} \cdot m_{C'}$ is minus the Welschinger multiplicity.

Case 3: V is a vertex of type (D1) or (D2) (we are then in case (c) of the theorem). Remember from lemma 5.11 that V can then be resolved into a vertex of type (2) to (4) and a vertex of type (9). The vertex of type (7) corresponding to the vertex of type (9) has to lie behind one of the even edges at the 4-valent vertex by definition 5.2 (b); we choose it to be behind the edge with direction v_2. The orientation and the parity of the bounded edge which appears when resolving are determined.

Observe that resolution I does not exist for the vertex of type (D1) as the 3-valent vertices that appear then are not allowed for bridge curves. The vertices appearing are listed in the table below. The last column $m_{\mathrm{I/II/III}}$ shows the absolute value of the product of the two vertex multiplicities in the resolutions I, II, and III.

codim-1	resolution I			resolution II			resolution III		
case	V	W	m_{I}	V	W	m_{II}	V	W	m_{III}
D1				(2)	(9)	1	(2)	(9)	1
D2	(4)	(9)	$|(v_1,v_2)|$	(3)	(9)	$|(v_1,v_3)|$	(3)	(9)	$|(v_1,v_4)|$

We have to check if conditions (b) and (c) of definition 5.2 are satisfied. Connections between vertices of type (7) to vertices of type (8) are not modified as no vertices of type (7), (8) and markings are involved in the resolutions. Similarly, the connection between the vertex of type (9) and the corresponding vertex of type (7) is not modified as the vertex of type (7) lies behind the edge of direction v_2. Hence, condition (b) is satisfied in all resolutions or in none of them. As no markings are involved in the resolutions, the set M, the splitting of M, and the total order are also preserved. So condition (c) holds in all three resolutions or in none of them.

In order to prove the local invariance we also have to compute the direction of the string movement as in definition 5.13. In resolution I we create a vertex of type (9), so the edge E of definition 5.13 becomes longer.

As in the proof of theorem 3.6 we can imagine to have for the other resolutions II and III two other markings $P_1, P_2 \in \mathbb{R}^2$ on the edges v_1, v_2 as these are fixed. Hence we have two bounded edges of lengths l_1 and l_2, in addition to the (by resolving) new inserted bounded edge of length a. The direction of the string movement as in definition 5.13 is positive if and only if l_2 becomes longer when a becomes longer. We can describe the condition that the curve has to pass through the given point conditions by the following linear systems of equations in the variables l_1, l_2, a.

	II		
l_1	l_2	a	
$-v_1$	v_2	$-v_1-v_3$	P_2-P_1

	III		
l_1	l_2	a	
$-v_1$	v_2	$-v_1-v_4$	P_2-P_1

Obviously, these systems both have a one-dimensional space of solutions. In case II the homogeneous solution vector (l_1, l_2, a) has the following entries:

$$l_1 = (v_2, -v_1-v_3), \qquad l_2 = -(-v_1, -v_1-v_3), \qquad a = (-v_1, v_2),$$

where as above (v_i, v_j) is the determinant of the matrix consisting of the column vectors v_i, v_j. So in order to determine the direction of the string movement we have to multiply the signs of l_2 and a, that is $\mathrm{sign}(v_1, v_3)\,\mathrm{sign}(v_1, v_2)$. In case III we just have to substitute the vector v_3 by v_4 and

obtain therefore as sign $\text{sign}(v_1, v_4)\,\text{sign}(v_1, v_2)$. So in total the sign for the directions of the string movements are given by the following table.

	sign for I	sign for II	sign for III
(D)	1	$\text{sign}((v_1, v_3)(v_1, v_2))$	$\text{sign}((v_1, v_4)(v_1, v_2))$

We are now able to verify the local invariance. We will use the same identities to deal with vertex multiplicities and signs as in the proof of theorem 3.6. Mainly, we use the formulas $\text{sign}(v_i, v_j)\, i^{|(v_i, v_j)|-1} = i^{(v_i, v_j)-1}$ if $|(v_i, v_j)|$ is odd and $i^{|(v_i, v_j)|-1} = i^{(v_i, v_j)-1}$ if $|(v_i, v_j)|$ is even.

In case (D1), we then obtain for the product of the vertex multiplicities together with the direction of the string movement in the resolutions II and III:

$$(\text{II}) = \text{sign}((v_1, v_3)(v_1, v_2)) \cdot i^{|(v_1, v_3)|-1} \cdot i^{|(v_2, v_4)|-1} = \text{sign}(v_1, v_2) \cdot i^{(v_1, v_3)+(v_4, v_2)-2},$$

$$(\text{III}) = \text{sign}((v_1, v_4)(v_1, v_2)) \cdot i^{|(v_1, v_4)|-1} \cdot i^{|(v_2, v_3)|-1} = \text{sign}(v_1, v_2) \cdot i^{(v_1, v_4)+(v_2, v_3)-2}.$$

We have $\text{sign}(v_1, v_2) \neq 0$ since v_1 and v_2 cannot be parallel as our curves pass through conditions in general position. Dividing equation (III) by (II) yields $i^{2(v_3, v_1)} = (-1)^{(v_3, v_1)} = -1$ as (v_3, v_1) is odd. Hence (II)+(III)= 0.

Similarly, for (D2) we obtain:

$$(\text{I}) = |(v_1, v_2)| \cdot i^{|(v_1, v_2)|-1} \cdot i^{|(v_3, v_4)|-1} = \text{sign}(v_1, v_2) \cdot (v_1, v_2)\, i^{(v_1, v_2)+(v_3, v_4)-2},$$

$$(\text{II}) = \text{sign}((v_1, v_3)(v_1, v_2)) \cdot |(v_1, v_3))| \cdot i^{|(v_1, v_3)|-1} \cdot i^{|(v_2, v_4)|} = \text{sign}(v_1, v_2) \cdot (v_1, v_3)\, i^{(v_1, v_3)+(v_4, v_2)-2},$$

$$(\text{III}) = \text{sign}((v_1, v_4)(v_1, v_2)) \cdot |(v_1, v_4)| \cdot i^{|(v_1, v_4)|-1} \cdot i^{|(v_2, v_3)|-1} = \text{sign}(v_1, v_2) \cdot (v_1, v_4)\, i^{(v_1, v_4)+(v_2, v_3)-2}.$$

Let us divide all three terms by $\text{sign}(v_1, v_2)\, i^{(v_1, v_2)+(v_3, v_4)-2}$. For (I) we then get (v_1, v_2). In term (II) we obtain $i^{2(v_2, v_1)} \cdot (v_1, v_3) = (-1)^{(v_2, v_1)} \cdot (v_1, v_3) = (v_1, v_3)$ as (v_2, v_1) is even. Finally, for (III) we get $i^{2(v_1, v_4)} \cdot (v_1, v_4) = (-1)^{(v_1, v_4)} \cdot (v_1, v_4) = (v_1, v_4)$ as (v_1, v_4) is also even. So we have (I)+(II)+(III)= $(v_1, v_2) + (v_1, v_3) + (v_1, v_4) = 0$.
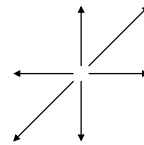
Hence we have shown the invariance for all codimension-1 cases for bridge curves. $\qquad\square$

In order to prove the equality of broccoli and Welschinger numbers with the idea of remark 5.12 we need one more final ingredient: that each edge in the bridge graph is actually bounded, i.e. that the string movement in each 1-dimensional type of bridge curves is bounded in both directions by a codimension-1 case. It is actually only this last step that requires a toric Del Pezzo degree and thus spoils the equality of broccoli and Welschinger numbers (as well as the invariance of Welschinger numbers, see example 4.25) in other cases.

Lemma 5.15 (Boundedness of bridges)**.** *Assume that Δ is a toric Del Pezzo degree (see definition 4.22). Let C be a bridge curve through $\mathscr{P}$ with a vertex of type (9), thus having a string as in remark 5.10. Then the movement of the string within this combinatorial type is bounded in both directions.*

Proof. Assume that we have a bridge curve through $\mathscr{P}$ with a string that can be moved infinitely far. By the proof of proposition 5.1 in [GM08] such a string then has to consist of two edges which are both ends of the curve.

As we are dealing with bridge curves the string must then consist of the two odd edges adjacent to the vertex of type (9). From the definition of the vertex type (9) we know that the two ends cannot have the same direction. Considering definition 4.22 of toric Del Pezzo degrees we thus see that these ends have two of the directions shown in the picture on the right. But in all these cases the third direction at the vertex of type (9) would be odd (in contradiction to the definition of type (9)) or 0 (which is impossible for curves through conditions in general position). Hence the string movement cannot be unbounded.

$\square$

Corollary 5.16 (Welschinger numbers = broccoli invariants in the toric Del Pezzo case)**.** *Let $r, s \geq 0$, let $\Delta = (v_1, \ldots, v_n)$ be a toric Del Pezzo degree, and let $F \subset \{1, \ldots, n\}$ such that $|\Delta| - 1 = r + 2s + |F|$. Fix a configuration $\mathscr{P}$ of conditions in general position. Then $N^W_{(r,s)}(\Delta, F, \mathscr{P}) = N^B_{(r,s)}(\Delta, F, \mathscr{P})$.*

Proof. By theorem 5.14 and definitions 3.5 and 4.13 we have

$$|G(\Delta, F)| \cdot \left(N^B_{(r,s)}(\Delta, F, \mathscr{P}) - N^W_{(r,s)}(\Delta, F, \mathscr{P}) \right) = \sum_C \sum_{C'} \operatorname{sign}_{C'} \cdot m_{C'},$$

where the sum is taken over all C as in theorem 5.14 and all resolutions C' of C (i.e. over all vertices and adjacent edges in the bridge graph of remark 5.12). Note that this in fact a finite sum since there are only finitely many types of bridge curves. Now by lemma 5.15 each 1-dimensional type C' of bridge curves occurs in this sum exactly twice with the same multiplicity, once with a positive and once with a negative sign. Hence the sum is 0, proving the corollary. $\square$

Corollary 5.17 (Invariance of Welschinger numbers in the toric Del Pezzo case)**.** *With the assumptions and notations as in corollary 5.16, the Welschinger numbers $N^W_{(r,s)}(\Delta, F, \mathscr{P})$ are independent of the conditions $\mathscr{P}$.*
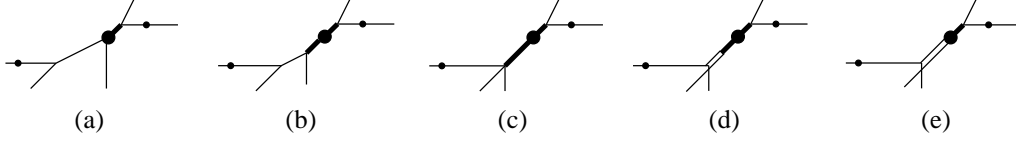
Proof. This follows from corollary 5.16 and theorem 3.6. $\square$

In the remaining part of this section we want to construct bridges explicitly and give some examples. The following algorithm, which follows from the proof of theorem 5.14, shows how to construct a bridge from a given starting point.

Algorithm 5.18 (Bridge algorithm)**.** Let $r, s \geq 0$, let $\Delta = (v_1, \ldots, v_n)$ be a toric Del Pezzo degree, and let $F \subset \{1, \ldots, n\}$ be such that $|\Delta| - 1 = r + 2s + |F|$. Fix a configuration $\mathscr{P}$ of conditions in general position. Consider a bridge curve C passing through $\mathscr{P}$; we want to construct the bridge that contains C.

(1) If C is a broccoli and Welschinger curve simultaneously (hence $M_{(8)} = M_{(9)} = M_{(6a)} = \emptyset$), do nothing.

(2) Given a bridge curve C with $M_{(9)} \neq \emptyset$ (hence with a string) together with a direction for the movement of the string, move the string in the direction until we hit a codimension-1 type C' as in lemma 5.11. Go to (2) with each new resolution in the direction away from C'.

(3) If the curve is a broccoli curve, that is $M_{(8)} = M_{(9)} = \emptyset$, choose the smallest vertex in $M_{(6a)}$ under the total order defined in 5.2 (c). Pull out an even edge of this vertex of type (6a) in order to create a vertex of type (7) and a vertex of type (9), thus producing a bridge curve with a string and a direction for the movement. Go to (2).

(4) If the curve is a Welschinger curve, that is $M_{(9)} = M_{(6a)} = \emptyset$, choose the vertex of type (8) corresponding to the biggest vertex in $M_{(8)}$ under the total order defined in 5.2 (c). Pull apart the two odd edges in order to create a string between the two even edges and a direction for the movement. We thus transform the vertex of type (8) into a vertex of type (3) and a vertex of type (9). Go to (2).

(5) If the curve is a bridge curve with $M_{(9)} = \emptyset$, but $M_{(8)} \neq \emptyset \neq M_{(6a)}$, we can choose the biggest vertex (under the total order) in $M_{(8)}$ or the smallest in $M_{(6a)}$ in order to construct the bridge in direction "broccoli" or in direction "Welschinger". Transform the vertex as described in the two last items, respectively, thus producing a bridge curve with a string and a direction. Go to (2).
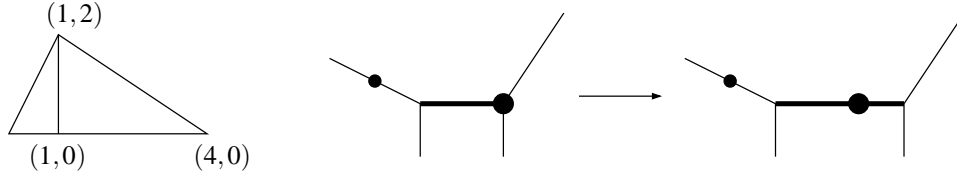
Example 5.19 (A bridge connecting only broccoli curves). Following algorithm 5.18, the following picture shows a bridge connecting one broccoli curve (a) to another broccoli curve (e) (and to no Welschinger curve). In curve (c) we resolve a 4-valent vertex of type (D1). The types (b) and (d) are 1-dimensional, the other three 0-dimensional.
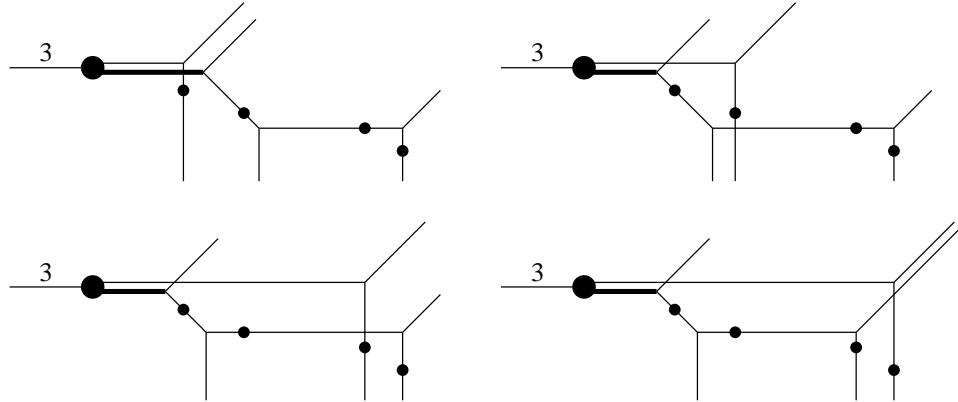

(a)   (b)   (c)   (d)   (e)

An example of a bridge connecting a broccoli curve with a Welschinger curve can be found in section 1.2 of the introduction.

Example 5.20 (Two cases that are not toric Del Pezzo). The boundedness of bridges of lemma 5.15, and consequently the equality of broccoli and Welschinger numbers as well as the invariance of Welschinger numbers, are false in general for degrees that are not toric Del Pezzo:

(a) Consider the following Newton polytope and its subdivision. It is obviously not toric Del Pezzo. A broccoli curve having this Newton subdivision is depicted on the right hand side. Starting the bridge as in algorithm 5.18 yields a string going to infinity (very right hand side), so the broccoli curve is not connected to a Welschinger curve by a bridge.

(b) Recall example 4.25 where we have shown that Welschinger numbers are not invariant if we do not have a toric Del Pezzo degree. If we choose the point configuration $\mathscr{P}$ as in example 4.25, the Welschinger curves C_1, C_2, C_3 with multiplicity 3 shown there are also broccoli curves, and in addition there are 4 more broccoli curves passing through $\mathscr{P}$ as depicted below.

Each of them has multiplicity -2, so the broccoli invariant is $N^B_{(r,s)}(\Delta, \mathscr{P}) = 3 \cdot 3 + 4 \cdot (-2) = 1$. In particular, it is not equal to $N^W_{(r,s)}(\Delta, \mathscr{P}) = 9$. Indeed, starting a bridge at the complex marking of each of the four curves above gives a curve having a string going to infinity as in (a), so the contribution of -8 to the broccoli invariant is not seen on the Welschinger side.

Example 5.21 (Broccoli invariants for degrees with non-fixed even ends). By remark 5.10 the ends of a string are always unfixed and odd. In particular, this means that the proof of lemma 5.15 (and thus also of the equality of broccoli and Welschinger numbers) only requires that the *unfixed odd* ends in Δ are those occurring in a toric Del Pezzo degree.

Let us review example 4.14 from this point of view.

(a) If Δ has more than one non-fixed even end, and all other non-fixed ends are only those occurring in a toric Del Pezzo degree, then the result $N_{(r,s)}^W(\Delta, F, \mathscr{P}) = 0$ of example 4.14 (a) implies that also $N_{(r,s)}^B(\Delta, F) = 0$.

(b) If Δ has one non-fixed even end, and all other ends are non-fixed and among those occurring in a toric Del Pezzo degree, then the formula for $N_{(r,s)}^W(\Delta, \mathscr{P})$ of example 4.14 (b) holds in the same way for $N_{(r,s)}^B(\Delta)$.

6. THE CAPORASO-HARRIS FORMULA FOR BROCCOLI CURVES

In this section, we establish a Caporaso-Harris formula for broccoli curves of degree dual to the triangle with endpoints $(0,0)$, $(d,0)$ and $(0,d)$. This is a recursive formula computing all broccoli invariants with weight conditions on fixed and non-fixed left ends in addition to the usual point conditions. As usual for Caporaso-Harris type formulas, the idea to obtain these relations is to move one of the point conditions to the far left so that the curve splits into a left part (passing through the moved point) and a right part (passing through the remaining points). Since broccoli invariants of curves with ends of weight one (i.e. of degree d) equal Welschinger numbers $N_{(r,s)}^W(d)$ by corollary 5.16 and the latter equal Welschinger invariants $W_{\mathbb{P}^2}(d, r, s)$ by the Correspondence Theorem 4.23, our formula then computes all Welschinger invariants of the plane recursively.

It is also possible to use Welschinger curves directly to establish a similar formula. However, since the numbers of Welschinger curves of degree dual to the triangle with endpoints $(0,0)$, $(d,0)$, and $(0,d)$ and with ends of higher weight are not invariant (as we have seen in example 4.25), the arguments are then getting significantly more complicated as one always has to pick special configurations of points. This is the content of [ABLdM10]. There, the authors pick a configuration of points such that the Welschinger curves passing through these points decompose totally into floors (see proposition 6.8), and count them by means of floor diagrams. This yields a recursive formula for floor diagrams which also computes all Welschinger invariants of the plane.

Let us first fix some notation.

Notation 6.1. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\beta = (\beta_1, \ldots, \beta_{m'})$, $\alpha^1 = (\alpha_1^1, \ldots, \alpha_{m_1}^1), \ldots, \alpha^k = (\alpha_1^k, \ldots, \alpha_{m_k}^k)$ be finite sequences with $\alpha_i, \beta_i, \alpha_i^j \in \mathbb{N}$. For simplicity, we will usually consider them to be infinite sequences by setting the remaining entries to 0. We then define:

(a) $|\alpha| := \sum_{i=1}^m \alpha_i$,

(b) $I\alpha := \sum_{i=1}^m i \cdot \alpha_i$,

(c) $\alpha + \beta := (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots)$,

(d) $\alpha \leq \beta :\Leftrightarrow \alpha_i \leq \beta_i$ for all i,

(e) $\alpha < \beta :\Leftrightarrow \alpha \leq \beta$ and $\alpha \neq \beta$,

(f) $\binom{n}{\alpha_1, \ldots, \alpha_m} := \frac{n!}{\alpha_1! \cdot \ldots \cdot \alpha_m! (n - \alpha_1 - \ldots - \alpha_m)!}$ for $|\alpha| \leq n$,

(g) $\binom{\alpha}{\alpha^1, \ldots, \alpha^k} := \prod_i \binom{\alpha_i}{\alpha_i^1, \ldots, \alpha_i^k}$.

Furthermore, we define e_k to be the sequence having only 0 as entries except a 1 in the k-th entry.

Definition 6.2 (Broccoli curves of type (α,β))**.** Let $d > 0$, and let α and β be two sequences satisfying $I\alpha + I\beta = d$. We define $\Delta(\alpha,\beta)$ to be the degree consisting of d times the vectors $(0,-1)$ and $(1,1)$ each, and $\alpha_i + \beta_i$ times $(-i,0)$ for all i (in any fixed order). Let $F(\alpha,\beta) \subset \{1,\dots,|\Delta(\alpha,\beta)|\}$ be a fixed subset with $|\alpha|$ elements such that the entries of $\Delta(\alpha,\beta)$ with index in F are α_i times $(-i,0)$ for all i. If no confusion can result we will often abbreviate $\Delta(\alpha,\beta)$ as Δ and $F(\alpha,\beta)$ as F.

Broccoli curves in $M^B_{(r,s)}(\Delta,F)$ will be called *curves of type* (α,β). We speak of their unmarked ends with directions $(-i,0)$ as the *left ends*. So α_i and β_i are the numbers of fixed and non-fixed left ends of weight i, respectively.

Definition 6.3 (Relative broccoli invariants)**.** Let $\Delta = \Delta(\alpha,\beta)$ and $F = F(\alpha,\beta)$ be as in definition 6.2, and r,s such that the dimension condition $|\Delta| - 1 - |F| = 2d + |\beta| - 1 = r + 2s$ is satisfied. To simplify notation, we define the *relative broccoli invariant*

$$N^d(\alpha,\beta,s) := N^B_{(r,s)}(\Delta(\alpha,\beta),F(\alpha,\beta)).$$

Remark 6.4 (Unlabeled non-fixed ends)**.** Notice that by remark 4.18 a broccoli curve without labels on the unmarked ends yields $2^{-k} \cdot |G(\Delta,F)|$ labeled curves contributing to the broccoli invariant, where $|G(\Delta,F)|$ as in definition 2.8 (b) denotes the number of ways to relabel the non-fixed unmarked ends without changing the degree, and $k = n_{(6b)} + n_{(8)}$ is the number of double ends. In contrast, in the definition 3.5 of broccoli invariants we multiply the number of broccoli curves with $\frac{1}{|G(\Delta,F)|}$. Thus a curve without labels contributes 2^{-k} to the count. Hence, when counting broccoli curves whose non-fixed unmarked ends are not labeled, we have to change the multiplicity of vertices of type (6b) to $\frac{1}{2} \cdot i^{-1}$. In the following, we will drop the labels of the non-fixed ends and change the multiplicity accordingly. Note that for the degree Δ and F as above we have $|G(\Delta,F)| = d! \cdot d! \cdot \beta_1! \cdot \beta_2! \cdot \cdots$.

Remark 6.5**.** It follows from theorem 3.6 that $N^d(\alpha,\beta,s)$ is invariant, i.e. does not depend on the choice of the conditions. Note that if $\alpha = (0)$ and $\beta = (d)$ then

$$N^d((0),(d),s) = N^B_{(r,s)}(d) = N^W_{(r,s)}(d) = W_{\mathbb{P}^2}(d, 3d - 2s - 1, s),$$

where the second equality follows from theorem 5.14 and the last equality from theorem 4.23.

Now we describe the properties of configurations $\mathscr{P}$ of points that we obtain by moving one of the point conditions (w.l.o.g. P_1) to the left. Let us show first that then curves satisfying these conditions decompose into a left and a right part.

Lemma 6.6 (Decomposing curves into a left and right part)**.** *Let Δ and F be as in definition 6.2, and let $2d + |\beta| - 1 = r + 2s$. Fix a small real number $\varepsilon > 0$ and a large one $N > 0$. Choose $r + s$ (real and complex) points $P_1,\dots,P_{r+s}$ and $|\alpha|$ y-coordinates for the fixed left ends in general position such that*

- *the y-coordinates of all P_i and the fixed ends are in the open interval $(-\varepsilon,\varepsilon)$,*
- *the x-coordinates of $P_2,\dots,P_{r+s}$ are in $(-\varepsilon,\varepsilon)$,*
- *the x-coordinate of P_1 is smaller than $-N$.*

Let $C = (\Gamma, x_1,\dots,x_{r+s}, y_1,\dots,y_n, h) \in M^B_{(r,s)}(\Delta,F)$ be a broccoli curve satisfying these conditions. Then no vertex of C can have its y-coordinate below $-\varepsilon$ or above ε. There is a rectangle $R = [a,b] \times [-\varepsilon,\varepsilon]$ (with $a \geq -N$, $b \leq -\varepsilon$ only depending on d) such that $R \cap h(\Gamma)$ contains only horizontal edges of C.

Proof. Notice that it follows from lemma 2.13 that each connected component of C minus the marked points contains exactly one non-fixed unmarked end, a statement analogous to remark 2.10 of [GM07a]. The fact that the y-coordinates of the vertices of C cannot be above ε or below $-\varepsilon$ and the existence of the rectangle R follow analogously to the first part of the proof of theorem 4.3 of [GM07a]. $\square$

A configuration of points and y-coordinates for the fixed left ends as in lemma 6.6 can be obtained from any other by moving P_1 far to the left. So in this situation the curves decompose into a left and a right part connected by only horizontal edges in the rectangle R. A picture showing this can be found in example 6.9. In the following, we study the possibilities for the shapes of the left and right part.

Notation 6.7 (Left and right parts). With notations as in lemma 6.6, cut C at each bounded edge e such that $h(e) \cap R \neq \emptyset$. Denote the component passing through P_1 by C_0 (the left part), and the union of the other connected components by $\tilde{C}$ (the right part).

Proposition 6.8 (Possible shapes of the left and right part). *Let C_0 and $\tilde{C}$ be the left and right part of a broccoli curve as in lemma 6.6 and notation 6.7.*

(a) *If C_0 has no bounded edges, it looks like (A), (B), or (C) in the picture below (in which the edges are labeled with their weights). Moreover:*
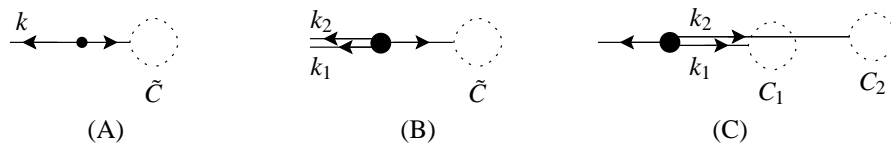
- *In case (A), $\tilde{C}$ is an irreducible curve of type $(\alpha + e_k, \beta - e_k)$.*

- *In case (B), $\tilde{C}$ is an irreducible curve of type $(\alpha + e_{k_1+k_2}, \beta - e_{k_1} - e_{k_2})$.*

- *In case (C), $\tilde{C}$ decomposes into two connected components C_1 and C_2 of types (α^1, β^1) resp. (α^2, β^2) with $I\alpha^j + I\beta^j = d_j$ for $j = 1, 2$, $d_1 + d_2 = d$, $\alpha^1 + \alpha^2 = \alpha + e_{k_1} + e_{k_2}$, and $\beta^1 + \beta^2 = \beta - e_{k_1+k_2}$. The curve C_j for $j = 1, 2$ passes through r_j real and s_j complex given points, where $2d_j + |\beta^j| - 1 = r_j + 2s_j$.*
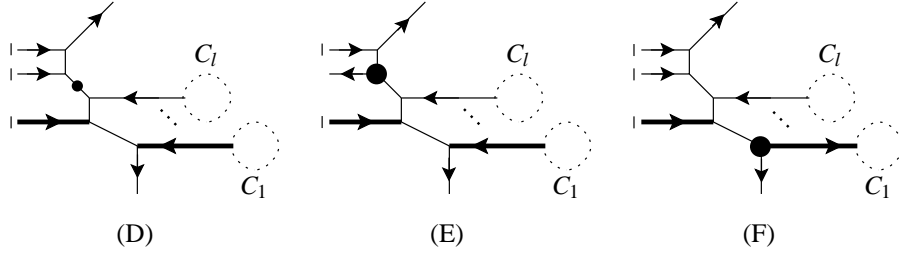
In case (A) (for real P_1) the left end is odd, in the cases (B) and (C) (for complex P_1) exactly one of the three edges adjacent to P_1 is even.

(b) *If C_0 has bounded edges (it is then called a floor), it looks like (D), (E), or (F) in the picture below, and has one end of direction $(0, -1)$ and one of direction $(1, 1)$. We call the ends of C_0 of direction $(i, 0)$ for $i > 0$ the right ends. Moreover:*

- *In case (D) (for real P_1), C_0 has only fixed left and right ends.*

- *In case (E) (for complex P_1), P_1 is adjacent to a left non-fixed end of C_0, and all other left and right ends of C_0 are fixed.*

- *In case (F) (for complex P_1), P_1 is adjacent to a right non-fixed end of C_0, and all other left and right ends of C_0 are fixed.*

In any case, $\tilde{C}$ consists of some number l of connected components $C_1, \ldots, C_l$. Each C_j is a curve of some type (α^j, β^j) with $I\alpha^j + I\beta^j = d_j$ and $\sum_{j=1}^l d_j = d - 1$. The curve C_j for $j = 1, \ldots, l$ passes through r_j real and s_j complex given points, where $2d_j + |\beta^j| - 1 = r_j + 2s_j$. Note that (D), (E), and (F) are meant to be schematic pictures in which the thin and thick horizontal edges are just examples. The non-horizontal edges are always odd however.

(A)　　　　　　(B)　　　　　　(C)

(D)   (E)   (F)

Proof. (a) Assume C_0 contains no bounded edge and P_1 is real. Then C_0 contains exactly one vertex, of type (1). Both adjacent edges are ends of C_0. Since C is connected, one of the ends of C_0 results from cutting a bounded horizontal edge of C. Because of the balancing condition, it follows that the other end is of direction $(-k, 0)$ for some $k > 0$, which has to be odd since P_1 is of vertex type (1). Hence we are then in case (A).

Assume now that P_1 is complex. Then C_0 consists of a vertex of type (5) or (6). At least one of the adjacent edges is of direction $(k, 0)$ for some $k > 0$ since it results from cutting a horizontal bounded edge. The other adjacent edges are ends of C. It follows from the balancing condition that all three adjacent edges are horizontal, and so we have type (B) or (C). Exactly one of the adjacent edges is even (and so vertex type (5) is impossible). In (A) and (B), we just cut one edge, so it follows that $\tilde{C}$ is irreducible and of the degree as claimed above. In (C), we cut two edges, so $\tilde{C}$ consists of two connected components C_1 and C_2. Ends of C_1 and C_2 are either ends of C or the two cut edges. Denote their weights by k_1 resp. k_2, then it follows that C_j is of a type (α^j, β^j) for $j = 1, 2$ with $\alpha^1 + \alpha^2 = \alpha + e_{k_1} + e_{k_2}$ and $\beta^1 + \beta^2 = \beta - e_{k_1 + k_2}$. If $2d_j + |\beta^j| - 1 < r_j + 2s_j$ for $j = 1$ or $j = 2$, then it follows that there is a connected component of Γ minus the marked ends which does not contain a non-fixed unmarked end, a contradiction to lemma 2.13. Thus we have $2d_j + |\beta^j| - 1 \geq r_j + 2s_j$, and since $2d_1 + |\beta^1| - 1 + 2d_2 + |\beta^2| - 1 = 2d + |\beta| - 3 = r + 2(s - 1) = r_1 + 2s_1 + r_2 + 2s_2$ it follows that $2d_j + |\beta^j| - 1 = r_j + 2s_j$ for $j = 1, 2$.
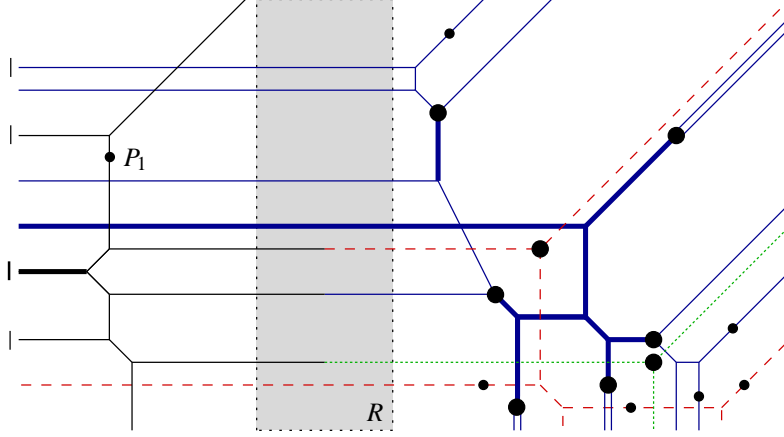
(b) Now assume that C_0 contains a bounded edge. By lemma 2.13, each connected component of C minus the marked points contains exactly one non-fixed unmarked end. If P_1 is real, removing the marked end x_1 satisfying $h(x_1) = P_1$ from Γ produces 2 connected components; if it is complex it produces 3 connected components. It follows that C_0 contains at most 2 non-fixed ends of C if P_1 is real, or 3 if P_1 is complex. Ends of C_0 are of direction $(k, 0)$ for some k (resulting from cutting horizontal bounded edges of C) or ends of C. If C_0 contains a bounded edge then C_0 cannot lie entirely in a horizontal line, since otherwise the length of such a bounded edge could not be fixed by our conditions. It follows by the balancing condition that C_0 must have ends of direction $(0, -1)$ and $(1, 1)$, and in fact an equal number of them. But since ends of these directions are non-fixed and we have at most 3 non-fixed ends of C in C_0, we conclude that there is exactly one end of direction $(0, -1)$ and $(1, 1)$ each. Since all other ends of C_0 are horizontal, it follows from the balancing condition that the directions of the bounded edges of C_0 are $\pm(a, 1)$ for some a. In particular, they are all odd.

If P_1 is real, C_0 cannot have more non-fixed ends of C than the two ends of direction $(0, -1)$ and $(1, 1)$. So then all left and right ends of C_0 are fixed, and we are in case (D). If P_1 is complex, there can be one non-fixed left end of C_0, which then has to be adjacent to P_1 as in case (E). Otherwise, P_1 has to be adjacent to a horizontal edge connecting C_0 with $\tilde{C}$. This is true because by the directions of the ends of C_0 and the balancing condition we can conclude that every vertex of C_0 is adjacent to an edge of direction $(k, 0)$ for some (positive or negative) k. Thus we are then in case (F).

Assume we have to cut l edges to produce C_0 and $\tilde{C}$, then $\tilde{C}$ consists of l connected components. Each connected component is a curve of some type (α^j, β^j) with $I\alpha^j + I\beta^j = d_j$. It follows from

the balancing condition that $\sum_{j=1}^{l} d_j = d - 1$. The equations $2d_j + |\beta^j| - 1 = r_j + 2s_j$ follow as in part (a). $\qquad\square$

Example 6.9. The picture shows an example of a curve C decomposing into a floor C_0 of type (D) on the left and a reducible curve $\tilde{C}$ on the right. C is of type $((3,1),(3,1))$ passing through $r = 7$ real and $s = 8$ complex points satisfying $2d + |\beta| - 1 = 20 + 4 - 1 = 23 = r + 2s$. We have chosen to move a real point to the left of the others.

The reducible curve $\tilde{C}$ consists of three connected components, C_1 (green dotted), C_2 (red dashed) and C_3 (blue solid). C_1 is a curve of type $((0),(1))$ passing through $s_1 = 1$ complex points, satisfying $2d_1 + |\beta^1| - 1 = 2 + 1 - 1 = 2 = r_1 + 2s_1$. C_2 is a curve of type $((0),(2))$ passing through $r_2 = 3$ real and $s_2 = 1$ complex points satisfying $2d_2 + |\beta^2| - 1 = 4 + 2 - 1 = 5 = r_2 + 2s_2$. C_3 is a curve of type $((1),(3,1))$ passing through $r_3 = 3$ real and $s_3 = 6$ complex points satisfying $2d_3 + |\beta^3| - 1 = 12 + 4 - 1 = 15 = r_3 + 2s_3$. We have $d_1 + d_2 + d_3 = 1 + 2 + 6 = d - 1$. All three curves are connected to C_0 via a horizontal edge of weight 1. We have $\beta = (3,1) = \beta^1 + \beta^2 + \beta^3 - 3e_1$ and $\alpha^1 + \alpha^2 + \alpha^3 = (1) < \alpha = (3,1)$.

Note that in the situation above there is always a unique possibility for C_0 once we are given the left and right ends of C_0 (together with their position for fixed ends) as well as the position of P_1. Thus, to determine $N^d(\alpha, \beta, s)$, we just have to determine the different contributions from all possibilities for $\tilde{C}$. This is the content of the following theorem.

Theorem 6.10 (Caporaso-Harris formula for $N^d(\alpha, \beta, s)$). *The following two recursive formulas hold for the invariants $N^d(\alpha, \beta, s)$, where we use the notation $r := 2d + |\beta| - 2s - 1$ (resp. $r_j := 2d_j + |\beta_j| - 2s_j - 1$ for all j) for the corresponding number of real markings in the invariant:*

(a) *(Moving a real point to the left) If $r > 0$ then*

$$N^d(\alpha, \beta, s) = \sum_{k \text{ odd}} N^d(\alpha + e_k, \beta - e_k, s) \tag{A}$$

$$+ \sum \frac{1}{l!} \binom{s}{s_1, \ldots, s_l} \binom{r-1}{r_1, \ldots, r_l} \binom{\alpha}{\alpha^1, \ldots, \alpha^l} \prod_{m \text{ even}} (-m)^{\alpha'_m} \prod_{\substack{j=1 \\ k_j \text{ even}}}^{l} k_j$$

$$\cdot \prod_{j=1}^{l} \left(\beta^j_{k_j} N^{d_j}(\alpha^j, \beta^j, s_j) \right) \tag{D}$$

where we set $\alpha' := \alpha - \sum_{j=1}^{l} \alpha^j$, and where the sum in (D) runs over all $l \geq 0$ and all $\alpha^j, \beta^j, k_j \geq 1, d_j \geq 1, s_j \geq 0$ for $1 \leq j \leq l$ satisfying $\sum_j \alpha^j < \alpha$, $\sum_j (\beta^j - e_{k_j}) = \beta$, $\sum_j d_j = d - 1$, $\sum_j s_j = s$.

(b) *(Moving a complex point to the left)* If $s > 0$ then

$$N^d(\alpha,\beta,s) = \sum -\frac{1}{2} N^d(\alpha + e_{k_1+k_2}, \beta - e_{k_1} - e_{k_2}, s-1) \tag{B}$$

$$+ \sum \frac{1}{2} \binom{s-1}{s_1,s_2} \binom{r}{r_1,r_2} \binom{\alpha}{\alpha^1,\alpha^2} \cdot \prod_{j=1}^{2} N^{d_j}(\alpha^j + e_{k_j}, \beta^j, s_j) \tag{C}$$

$$+ \sum \frac{1}{l!} \binom{s-1}{s_1,\ldots,s_l} \binom{r}{r_1,\ldots,r_l} \binom{\alpha}{\alpha^1,\ldots,\alpha^l} M_k \prod_{\substack{m \text{ even}}} (-m)^{\alpha'_m} \prod_{\substack{j=1 \\ k_j \text{ even}}}^{l} k_j$$

$$\cdot \prod_{j=1}^{l} \left(\beta^j_{k_j} N^{d_j}(\alpha^j, \beta^j, s_j) \right) \tag{E}$$

$$+ \sum \frac{1}{(l-1)!} \binom{s-1}{s_1,\ldots,s_l} \binom{r}{r_1,\ldots,r_l} \binom{\alpha}{\alpha^1,\ldots,\alpha^l} \tilde{M}_{k_1} \prod_{\substack{m \text{ even}}} (-m)^{\alpha'_m} \prod_{\substack{j=2 \\ k_j \text{ even}}}^{l} k_j$$

$$\cdot N^{d_1}(\alpha^1 + e_{k_1}, \beta^1, s_1) \prod_{j=2}^{l} \left(\beta^j_{k_j} N^{d_j}(\alpha^j, \beta^j, s_j) \right) \tag{F}$$

where as above $\alpha' := \alpha - \sum_{j=1}^{l} \alpha^j$, *and where the sums run over*

(B) *all* $k_1, k_2 \geq 1$ *such that at least one of them is odd;*

(C) *all* $\alpha^j, \beta^j, k_j \geq 1, d_j \geq 1, s_j \geq 0$ *for* $j \in \{1,2\}$ *such that at least one of* k_1, k_2 *is odd,* $\sum_j \alpha^j = \alpha$, $\sum_j \beta^j = \beta - e_{k_1+k_2}$, $\sum_j d_j = d$, $\sum_j s_j = s-1$.

(E) *all* $l \geq 0$ *and all* $\alpha^j, \beta^j, k \geq 1, k_j \geq 1, d_j \geq 1, s_j \geq 0$ *for* $1 \leq j \leq l$ *such that* $\sum_j \alpha^j < \alpha$, $\sum_j (\beta^j - e_{k_j}) = \beta - e_k$, $\sum_j d_j = d-1$, $\sum_j s_j = s-1$.

(F) *all* $l \geq 1$ *and all* $\alpha^j, \beta^j, k_j \geq 1, d_j \geq 1, s_j \geq 0$ *for* $1 \leq j \leq l$ *such that* $\sum_j \alpha^j < \alpha$, $\beta^1 + \sum_{j>1} (\beta^j - e_{k_j}) = \beta$, $\sum_j d_j = d-1$, $\sum_j s_j = s-1$.

Here, the numbers M_k *and* $\tilde{M}_k$ *are defined by*

$$M_k = \begin{cases} k & \text{if } k \text{ odd,} \\ -1 & \text{if } k \text{ even} \end{cases} \quad \text{and} \quad \tilde{M}_k = \begin{cases} k & \text{if } k \text{ odd,} \\ 1 & \text{if } k \text{ even.} \end{cases}$$

Of course, for both equations it is assumed that the sums are taken only over choices of variables such that all occurring sequences have only non-negative entries and all relative broccoli invariants satisfy the dimension condition.

Proof. As we have mentioned already we move one of the point conditions to the far left, so that each curve satisfying the conditions decomposes into a left part C_0 and a right part $\tilde{C}$. Since we have studied the possibilities for C_0 and $\tilde{C}$ in proposition 6.8 already it only remains to understand the different contributions to the relative broccoli invariant from each of these cases.

(a) The first formula arises from moving a real point to the left, so we have the cases (A) and (D).

(A) C_0 consists of one vertex of multiplicity 1, and $\tilde{C}$ has the same ends as C, with one odd non-fixed left end replaced by a fixed one. Thus we only have to sum over all possibilities of weights of this left end.

(D) We have to sum over all possibilities for $\tilde{C}$ to split into l connected components $C_1, \ldots, C_l$, where C_j is of type (α^j, β^j) with $I\alpha^j + I\beta^j = d_j$ and passes through r_j real and s_j complex points of $P_2, \ldots, P_{r+s}$. The right ends of C_0 are the gluing points for $C_1, \ldots, C_l$. They are fixed for C_0 and thus non-fixed for $C_1, \ldots, C_l$, i.e. they belong to $\beta^1, \ldots, \beta^l$. Let k_j be the

weight of the edge with which C_0 and C_j are connected. Then we have $\sum_{j=1}^{l}(\beta^j - e_{k_j}) = \beta$. Also, we have $\sum_{j=1}^{l} \alpha^j < \alpha$, and $\alpha' = \alpha - \sum_{j=1}^{l} \alpha^j$ is the sequence of fixed left ends adjacent to C_0. The multinomial coefficient $\binom{s}{s_1, \dots, s_l}$ gives the number of possibilities how the s complex points of $P_2, \dots, P_{r+s}$ can be distributed among the C_j. The second and third multinomial coefficient give the corresponding number for the real points and the fixed left ends, respectively.

It remains to take care of different multiplicity factors. First of all note that every fixed left end adjacent to C_0 (described by α') is not a fixed end of $\tilde{C}$ any more, so when counting the contribution from $\tilde{C}$ instead of C we lose a factor of i^{k-1} for every such end of weight k (remember that the weights of the ends of a curve C enter into the multiplicity m_C, see definition 2.17). Also, each such fixed end is adjacent to a vertex of C_0 whose multiplicity is $i^{k-1} \cdot k$ if k is even and i^{k-1} if k is odd. Thus, we lose a factor $i^{2k-2} = (-1)^{k-1} = 1$ if k is odd, and $k \cdot i^{2k-2} = k \cdot (-1)^{k-1} = -k$ if k is even. Therefore we have to multiply by $\prod_{m \text{ even}}(-m)^{\alpha'_m}$.

Similarly, for $j = 1, \dots, l$ the end of weight k_j with which C_j is connected to C_0 yields a factor of i^{k_j-1} in the multiplicity of $\tilde{C}$ that we do not need for C. The vertex of C_0 adjacent to such an edge has multiplicity $k_j \cdot i^{k_j-1}$ if k_j is even, and i^{k_j-1} if k_j is odd. Thus we need to multiply by $\prod_{j:k_j \text{ even}}^{l} k_j$.

The factors $\beta_{k_j}^{j}$ stand for the number of possibilities with which of the $\beta_{k_j}^{j}$ non-fixed ends of weight k_j the component C_j is connected to C_0. The factor $\frac{1}{l!}$ takes care of the overcounting due to the labeling of the components $C_1, \dots, C_l$. As C_0 has one end of direction $(0, -1)$ and $(1, 1)$ each it is clear that we must have $\sum_j d_j = d - 1$.

(b) In the second formula we move a complex point to the left, so we have four summands corresponding to the possibilities (B), (C), (E), and (F).

(B) We have to sum over all possibilities k_1 and k_2 for the weights of the two left ends which are adjacent to P_1. If we sum over all tuples (k_1, k_2), we overcount by a factor of 2 since these two weights are unordered. Therefore we multiply by $\frac{1}{2}$. For summands with $k_1 = k_2$, the $\frac{1}{2}$ takes care of the factor of $\frac{1}{2}$ in the multiplicity of the vertex of C_0 that we have to include when counting curves without labels at the unmarked ends (see remark 6.4). We lose factors of i^{k_1-1} and i^{k_2-1} since these two ends are not ends of $\tilde{C}$, and we lose a factor of i^{-1} for the vertex of C_0. Instead, we have a factor of $i^{k_1+k_2-1}$ for the end of $\tilde{C}$ with which it is glued to C_0. Thus, we have to multiply by -1.

(C) In this case we have to sum over all choices of the connecting weights k_1 and k_2 (which are fixed ends for C_1 and C_2), degrees d_1 and d_2, and numbers s_1 and s_2 of complex markings on each component. The symmetry factor $\frac{1}{2}$ cancels the overcounting due to the labeling of the two components. The binomial factors count the possibilities how the complex and real points and the fixed ends can be distributed among C_1 and C_2. In C_0, we have the left end contributing $i^{k_1+k_2-1}$ and a vertex contributing i^{-1}, in $\tilde{C}$ we have instead the two ends contributing i^{k_1-1} and i^{k_2-1}. So we do not need to multiply by a factor to take care of these multiplicities.

(E) The terms are essentially as in (D) above, except that in addition we have to sum over all possibilities for the weight k of the non-fixed left end adjacent to P_1. Also, this non-fixed end is not an end of any of the C_j, so the condition $\sum_j (\beta^j - e_{k_j}) = \beta$ has to be changed to $\sum_j (\beta^j - e_{k_j}) = \beta - e_k$. In addition to the factors of (D) we lose a factor of i^{k-1} for the end, and of i^{k-1} if k is even and $k \cdot i^{k-1}$ if k is odd for the vertex at P_1. So altogether we have to multiply by $i^{2k-2} = (-1)^{k-1} = -1$ if k is even and by k if k is odd.

(F) We get again a similar summand as in (E). However, here instead of summing over the possibilities for k we now have to choose one of the C_j — call it C_1 — which is adjacent to P_1. This component will then have an additional fixed end of weight k_1. So in the invariant for C_1 we have to replace α^1 by $\alpha^1 + e_{k_1}$; at the same time however we do not have to multiply this invariant by $\beta^1_{k_1}$ as C_1 is connected to C_0 by a fixed end. The fixed end of weight k_1 of C_1 contributes a factor of i^{k_1-1} to $\tilde{C}$. We lose the multiplicity of the vertex at P_1 which is i^{k_1-1} if k_1 is even and $k_1 \cdot i^{k_1-1}$ if k_1 is odd. Hence we have to multiply by $\tilde{M}_{k_1}$.

$\square$

Of course, theorem 6.10 now gives recursive formulas for all broccoli invariants $N^d(\alpha, \beta, s)$, and thus in particular by remark 6.5 also for the Welschinger numbers $W_{\mathbb{P}^2}(d, 3d-2s-1, s)$.

Example 6.11 (Relative broccoli invariants in degree 3). The following table shows all invariants $N^d(\alpha, \beta, s)$ for $d = 3$, as computed by theorem 6.10. The numbers in the last line are those that correspond to the degree-3 Welschinger invariants. The entries in the second last line are all 0 in accordance with example 4.14 (b).

α, β	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$
$(0,0,1),(0)$	3	1	-1		
$(0,1),(1)$	-12	-8	-4	0	
$(1,1),(0)$	-8	-4	0		
$(1),(0,1)$	0	0	0	0	
$(1),(2)$	8	6	4	2	
$(2),(1)$	8	6	4	2	
$(3),(0)$	6	4	2		
$(0),(0,0,1)$	3	1	-1	-3	
$(0),(1,1)$	0	0	0	0	
$(0),(3)$	8	6	4	2	0

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